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Mechanicam vero duplicem Veteres constituerunt: Rationalem quae per Demonstrationes accurate procedit, & Practicam. Ad practicam spectant Artes omnes Manuales, a quibus utique Mechanica nomen mutuata est. Cum autem Artifices parum accurate operari soleant, fit ut Mechanica omnis a Geometria ita distinguatur, ut quicquid accuratum sit ad Geometriam referatur, quicquid minus accuratum ad Mechanicam. Attamen errores non sunt Artis sed Artificum. Qui minus accurate operatur, imperfectior est Mechanicus, & si quis accuratissime operari posset, hic foret Mechanicus omnium perfectissimus.

NEWTON

La généralité que j'embrasse, au lieu d'éblouir nos lumières, nous découvrira plutôt les véritables loix de la Nature dans tout leur éclat, & on y trouvera des raisons encore plus fortes, d'en admirer la beauté & la simplicité.

EULER

Ceux qui aiment l'Analyse, verront avec plaisir la Méchanique en devenir une nouvelle branche ...

LAGRANGE

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Symmetry Relations for Orthotropic and Transversely Isotropic Materials

J. E. ADKINS

Communicated by A. E. GREEN

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1. Introduction

To describe mathematically the mechanical behaviour of a continuous medium it is first necessary to define tensors specifying the stress, the deformation and any required time rates of variation of these two quantities. These tensors must be defined in such a manner that the necessary requirements for invariance under rigid body motions are satisfied. This problem has been studied from various points of view by OLDROYD (1950), TRUESDELL (1955, a, b, c), RIVLIN & ERICKSEN (1955), NOLL (1955), GREEN & RIVLIN (1957) and others. The mechanical properties of the medium are then described by means of functional relations between the mechanical and kinematic tensors thus defined.

An important class of problems arises when one of the mechanical tensors is connected with the remainder by means of a polynomial relationship. To be explicit, we may suppose the stress tensor to be expressed as a polynomial in the elements of the $n+1$ tensors defining the deformation and its first n time rates of variation. The problem then reduces to the determination of a matrix polynomial suitably restricted to describe any symmetry properties of the material. When the body is isotropic the stress matrix may be expressed as the sum of a finite number of terms, each term being a matrix product formed from the kinematic matrices, multiplied by a scalar invariant function of these matrices. Each of these invariant functions may in turn be expressed as a polynomial in a finite number of invariants. This reduction has been carried out by RIVLIN (1955) for a polynomial in two matrices and by SPENCER & RIVLIN

(1959a, b) for the general case of n matrices. The writer (1958) has considered the corresponding problem for transversely isotropic and orthotropic bodies when two kinematic matrices are involved.

In the present paper, this work for aeolotropic materials is extended to the general case involving n matrices, attention being concentrated mainly on the transversely isotropic case which is treated from a slightly different viewpoint from that previously adopted. A preliminary reduction is achieved using results for the invariants of linear and quadratic forms under transformations of the orthogonal group (WEITZENBÖCK 1923, TURNBULL 1928). The results are simplified by using the matrix relations derived by SPENCER & RIVLIN and by further applications of the Cayley Hamilton theorem. For transversely isotropic bodies an alternative formulation is possible based on the decomposition of the 3×3 symmetric kinematic matrices into a system of symmetrical and unsymmetrical 2×2 matrices. This formulation is given in § 6 and relations between the two alternative systems of invariants for two matrices are derived in § 7. These relations are readily obtained by using the symbolic method of classical invariant theory, and by an extension of this method, given in § 8, the total number of coefficients and invariants required for the description of transverse isotropy may be considerably reduced. The stress strain relations for an elastic body resulting from the two formulations are given in § 9. The generalization to $n + 1$ matrices for orthotropic bodies is indicated in § 10.

2. Notation and formulae

We consider a body of homogeneous material each particle of which is at rest at time $t = 0$ and which subsequently undergoes a continuously varying deformation. A system of curvilinear coordinates θ^i is associated with elements of the material and moves with the body as it is deformed. The covariant and contravariant metric tensors for this system at the initial time $t = 0$ are $\gamma_{ij} = \gamma_{ij}(\theta^r, 0)$ and $\gamma^{ij} = \gamma^{ij}(\theta^r, 0)$ respectively, the corresponding quantities at the current time t being $\Gamma_{ij} = \Gamma_{ij}(\theta^r, t)$ and $\Gamma^{ij} = \Gamma^{ij}(\theta^r, t)$. In terms of these quantities the covariant strain tensor η_{ij} is defined by

$$\eta_{ij} = \frac{1}{2} (\Gamma_{ij} - \gamma_{ij}), \quad (2.1)$$

and by differentiation we may derive successive covariant rate of strain tensors $\alpha_{ij}^{(1)}, \alpha_{ij}^{(2)}, \dots, \alpha_{ij}^{(n)}$. Thus

$$\alpha_{ij}^{(r)} = \frac{D^r \eta_{ij}}{Dt^r} = \frac{1}{2} \frac{D^r \Gamma_{ij}}{Dt^r} \quad (r = 1, 2, \dots, n), \quad (2.2)$$

where D/Dt denotes differentiation with respect to t holding the convected coordinates θ^i constant. Expressions for these quantities referred to frames of reference fixed in space have been given by OLDROYD (1950) and others; RIVLIN & ERICKSEN (1955) have derived equivalent expressions for the description of the motion of a deforming material, again referred to a fixed frame of reference, their starting point being the displacement, velocity and successive acceleration gradients.

In the present work it is convenient to use the metric tensors γ^{ij}, γ_{ij} to raise and lower affixes. We therefore write

$$\eta_j^i = \gamma^{ik} \eta_{kj}, \quad \eta^{ij} = \gamma^{ik} \eta_k^j, \quad (2.3)$$

with corresponding expressions for the remaining kinematic tensors derived from (2.2). When the curvilinear system ϑ^i is such that it coincides with a rectangular Cartesian system $x^i (=x_i)$ in the undeformed body at time $t=0$ we denote the quantities $\eta_{ij}, \alpha_{ij}^{(r)}$ by $e_{ij}, a_{ij}^{(r)}$ respectively. With this choice of coordinates the metric tensors γ_{ij}, γ^{ij} reduce to Kronecker deltas δ_{ij}, δ^{ij} and the covariant, mixed and contravariant components derived by the process (2.3) are equal. From these components we form the symmetric kinematic matrices

$$\mathbf{E} = \|e_{ij}\|, \quad \mathbf{A}_r = \|a_{ij}^{(r)}\|, \quad (r = 1, 2, \dots, n). \quad (2.4)$$

The contravariant stress tensor referred to the convected system ϑ^i is denoted by τ^{ij} , or by t^{ij} if $\vartheta^i = x_i$ when $t=0$. A homogeneous, uniformly aeolotropic material may then be defined by the tensor relation

$$t^{ij} = f^{ij}(e_{rs}, a_{pq}^{(1)}, \dots, a_{uv}^{(n)}) \quad (f^{ij} = f^{ji}), \quad (2.5)$$

where the functions f^{ij} are polynomials in the arguments indicated. Any coefficients occurring in these polynomials are material constants which are independent of position throughout the material and do not depend upon the deformation. More generally, in place of (2.5) we may write

$$t^{ij} = \sum_{r=1}^m \varphi_r f_{(r)}^{ij}, \quad (2.6)$$

where $f_{(r)}^{ij}$ are symmetric polynomial tensor functions of the kinematic tensors $e_{pq}, a_{st}^{(k)}$ ($k=1, 2, \dots, n$) while the quantities φ_r are invariant functions of these tensors which are not necessarily polynomials but which are without singularities. By a suitable choice for the functions $\varphi_r, f_{(r)}^{ij}$, (2.6) may be made to yield the stress deformation relation for aeolotropic bodies derived from a different point of view by GREEN & RIVLIN (1957). We shall here be concerned mainly with the form of the functions $f_{(r)}^{ij}$ and shall therefore restrict attention to the simpler expression (2.5) from which, by a tensor transformation we obtain

$$\begin{aligned} \tau^{ij} &= \frac{\partial \vartheta^i}{\partial x^r} \frac{\partial \vartheta^j}{\partial x^s} t^{rs} \\ &= A_{rs}^{ij} t^{rs} = C_{rs}^{ij} t^{rs}, \end{aligned} \quad (2.7)$$

where

$$A_{rs}^{ij} = \frac{\partial \vartheta^i}{\partial x^r} \frac{\partial \vartheta^j}{\partial x^s}, \quad C_{rs}^{ij} = \frac{1}{2} (A_{rs}^{ij} + A_{sr}^{ij}). \quad (2.8)$$

Let $\mathbf{Q}$ be an arbitrary symmetric 3×3 matrix with elements q_{ij} referred to the coordinate system x_i . If we form the quantity

$$F(q_{ij}, e_{kl}, a_{st}^{(r)}) = q_{ij} t^{ij} = q_{ij} f^{ij}, \quad (2.9)$$

then F is an invariant polynomial function of the arguments $q_{ij}, e_{kl}, a_{st}^{(r)}$ which is linear and homogeneous in the elements q_{ij} . Also, F is symmetrical in the components q_{ij} and in the components of the kinematic tensors, so that interchange of any pair of suffixes of q_{ij} leaves the form of F unchanged. Furthermore, from (2.9) we have

$$t^{ij} = \frac{1}{2} \left(\frac{\partial F}{\partial q_{ij}} + \frac{\partial F}{\partial q_{ji}} \right) = f^{ij}. \quad (2.10)$$

Under the group of transformations (3.4) the forms (3.1) become

$$f_1 = \bar{a}_{ij} \bar{y}_i \bar{y}_j, \quad f_2 = \bar{b}_{ij} y_i y_j, \quad \dots, \quad f_m = \bar{m}_{ij} y_i y_j,$$

where the coefficients $\bar{a}_{ij}$ are given by

$$\bar{a}_{\lambda e} = M_{\lambda \eta} M_{e \nu} a_{\eta \nu}, \quad \bar{a}_{1\lambda} = M_{\lambda \eta} a_{1\eta}, \quad \bar{a}_{11} = a_{11}, \quad (3.6)$$

with similar expressions for $\bar{b}_{ij}, \dots, \bar{m}_{ij}$. For the symbols $a_i, a'_i, a''_i, \dots, m_i, m'_i, m''_i \dots$ we have the transformations

$$\bar{a}_1 = a_1, \quad \bar{a}_e = M_{e\sigma} a_\sigma, \quad (3.7)$$

corresponding to (3.4).

Combinations of the coefficients $a_{ij}, b_{ij} \dots m_{ij}$ which are invariant under the transformations (3.4) may be derived by adjoining to the quadratic forms (3.1) the linear form

$$l_i y_i = y_1, \quad (3.8)$$

where

$$(l_1, l_2, l_3) = (1, 0, 0). \quad (3.9)$$

It then follows (see, for example, WEITZENBÖCK 1923, chapter X) that any polynomial invariant formed from the coefficients $a_{ij}, b_{ij}, \dots, m_{ij}$ may be represented symbolically as a sum of products of the quantities

$$\begin{vmatrix} \xi_1 & \xi_2 & \xi_3 \\ \eta_1 & \eta_2 & \eta_3 \\ \zeta_1 & \zeta_2 & \zeta_3 \end{vmatrix} = (\xi \eta \zeta), \quad \xi_i \eta_i = (\xi \eta), \quad (3.10)$$

in which ξ_i, η_i, ζ_i may represent any of the symbols $a_i, a'_i, a''_i, \dots, m_i, m'_i, m''_i \dots$ or the set of coefficients l_i of the linear form (3.8). Alternatively, in the symbolic expression of any polynomial invariant, the system (3.10) may be replaced by

$$(\xi \eta l), \quad \xi_\alpha \eta_\alpha = (\xi | \eta), \quad \xi_i l_i = (\xi l) = \xi_1, \quad (3.11)$$

if we remember that

$$\begin{aligned} (\xi \eta) &= (\xi | \eta) + (\xi l) (\eta l), \\ (\xi \eta \zeta) &= (\xi \eta l) (l \zeta) - (\xi \zeta l) (l \eta) + (\eta \zeta l) (l \xi). \end{aligned} \quad (3.12)$$

If we consider polynomials in the coefficients of (3.1) which are invariant not only under transformations of the subgroup (3.4), but also with respect to the reflection

$$(\bar{y}_1, \bar{y}_2, \bar{y}_3) = (y_1, y_2, -y_3), \quad (3.13)$$

the determinants $(\xi \eta \zeta), (\xi \eta l)$ may be omitted from the symbolic representations (3.10), (3.11). For since such determinants change sign under the transformation (3.13), an even number of them must occur in each symbolic product, and the product of each pair of these determinants can be expressed as a polynomial in the scalar products $(\xi \eta)$ or $(\xi | \eta)$ and $(l \xi)$. It follows that any polynomial in the coefficients $a_{ij}, b_{ij}, \dots, m_{ij}$ which is invariant under the group of transformations defined by (3.4), (3.13) can be expressed symbolically as a polynomial in the scalar products

$$(\xi \eta) = \xi_i \eta_i, \quad (l \xi) = l_i \xi_i = \xi_1, \quad (3.14)$$

or as a polynomial in the products

$$(\xi|\eta) = \xi_\alpha \eta_\alpha, \quad (l\xi) = l_i \xi_i = \xi_1, \quad (3.15)$$

in which ξ, η may represent any of the symbols $a_i, a'_i, a''_i, \dots, m_i, m'_i, m''_i, \dots$. Evidently, if the polynomial basis is abandoned, the systems (3.14), (3.15) are sufficient to represent the invariants of the transformations (3.4) alone.

When the symbolic invariants are expressed in terms of the actual coefficients of the quadratic forms (3.1) it follows from (3.14) that any polynomial invariant formed from these coefficients can be expressed as a polynomial in the quantities

$$\begin{aligned} \lambda_{ij} \mu_{jk} \dots \xi_{lm} \nu_{mi}, & \quad (\text{i}) \\ \lambda_{1i} \mu_{ij} \dots \xi_{lm} \nu_{m1}, & \quad (\text{ii}) \end{aligned} \quad (3.16)$$

where $\lambda_{ij}, \mu_{ij}, \dots, \xi_{ij}, \nu_{ij}$ may represent any of the sets of coefficients $a_{ij}, b_{ij}, \dots, m_{ij}$ no restriction being placed, for the present, upon the number of times in which each of these coefficients may occur in the products (3.16). Alternatively, from (3.15) we may infer that any polynomial invariant may be expressed as a polynomial in the quantities

$$\begin{aligned} \lambda_{\alpha\beta} \mu_{\beta\gamma} \dots \xi_{\sigma\sigma} \nu_{\sigma\alpha}, & \quad (\text{i}) \\ \lambda_{1\alpha} \mu_{\alpha\beta} \dots \xi_{\sigma\sigma} \nu_{\sigma 1}. & \quad (\text{ii}) \end{aligned} \quad (3.17)$$

4. Transversely isotropic materials

A material is defined as transversely isotropic with respect to the x_1 -direction, if the relation (2.5), or the function F given by (2.9), remains form invariant under the group of transformations of the coordinates x_i in the undeformed body defined by

$$\begin{aligned} \bar{x}_1 &= x_1, & x_\sigma &= M_{\sigma\sigma} x_\sigma, \\ (\bar{x}_1, \bar{x}_2, \bar{x}_3) &= (x_1, x_2, \pm x_3), \end{aligned} \quad (4.1)$$

where the quantities $M_{\sigma\sigma}$ satisfy the orthogonality conditions (3.5). Since (4.1) combines the transformations (3.4), (3.13), and the tensors $q_{ij}, e_{ij}, a_{ij}^{(r)}, t^{ij}$ of § 2 behave under these transformations in the same way as the tensors $a_{ij}, b_{ij}, \dots, m_{ij}$ of § 3, we may infer that F may be expressed as a polynomial in the products (3.16) or (3.17) where $\lambda_{ij}, \mu_{ij}, \dots, \xi_{ij}, \nu_{ij}$ now represent elements of the matrices $\mathbf{Q}, \mathbf{E}, \mathbf{A}$, and the resulting products are linear in q_{ij} . We consider first the results which follow from the forms (3.16) and defer consideration of the alternative formulation (3.17) until § 6.

Remembering the definition (2.9) for F and the symmetry property (3.16) we may write

$$F = \frac{1}{2} \sum_{r=1}^N [({}^{(r)}P_{ij} q_{ij} {}^{(r)}Q_{j1} + {}^{(r)}P_{ij} q_{ij} {}^{(r)}Q_{i1}) \Phi_r + \frac{1}{2} \sum_{r=1}^M q_{ij} [{}^{(r)}R_{ij} + {}^{(r)}R_{ji}] \Psi_r. \quad (4.2)$$

In this expression, ${}^{(r)}P_{ij}, {}^{(r)}Q_{ij}, {}^{(r)}R_{ij}$ are elements of matrix products of the form

$$\mathbf{P} = \|\mathbf{P}_{ij}\|, \quad P_{ij} = b_{ik} c_{kl} \dots d_{mn} t_{nj}, \quad (4.3)$$

where $b_{ij}, c_{ij}, \dots, d_{ij}, f_{ij}$ may represent elements of any of the kinematic matrices $\mathbf{E}, \mathbf{A}$, or may be replaced by the Kronecker delta δ_{ij} ; special forms of these

coefficients are obtained by replacing one or more of ${}^{(r)}P_{ij}$, ${}^{(r)}Q_{ij}$ or ${}^{(r)}R_{ij}$ by δ_{ij} . The functions Φ_r , Ψ_r are polynomials in the invariants

$$\begin{aligned} I_r &= b_{ij} c_{jk} \dots d_{mn} f_{ni}, \\ J_r &= b_{1i} c_{ik} \dots d_{mn} f_{n1}. \end{aligned} \quad (4.4)$$

The quantities q_{ij} are absent from the products ${}^{(r)}P_{ij}$, ${}^{(r)}Q_{ij}$, ${}^{(r)}R_{ij}$ and the invariants I_r , J_r .

From (2.10) and (4.2) we obtain

$$t^{ij} = \frac{1}{2} \sum_{r=1}^N [{}^{(r)}P_{1i} {}^{(r)}Q_{j1} + {}^{(r)}P_{1j} {}^{(r)}Q_{i1}] \Phi_r + \frac{1}{2} \sum_{r=1}^M [{}^{(r)}R_{ij} + {}^{(r)}R_{ji}] \Psi_r, \quad (4.5)$$

and by a tensor transformation using (2.7) and (2.8)

$$\tau^{ij} = \sum_{r=1}^N {}^{(r)}P_{1p} C_{pq}^{ij} {}^{(r)}Q_{q1} \Phi_r + \sum_{r=1}^M C_{pq}^{ij} {}^{(r)}R_{pq} \Psi_r. \quad (4.6)$$

The invariants I_r and the coefficients of Ψ_r in (4.6) are the forms appropriate to isotropic bodies and may be expressed in more familiar forms in terms of the tensors η_{ij} , $\alpha_{ij}^{(r)}$ and the metric tensors Γ_{ij} , Γ^{ij} . These isotropic terms have been studied by SPENCER & RIVLIN (1959a, b) who have derived integrity bases for the invariants and have effected reductions of the symmetric matrix polynomials by using the Cayley Hamilton theorem and Peano's theorem. We are here concerned with the additional terms and invariants which arise from the assumption of transverse isotropy. In the expression for τ^{ij} the quantities C_{rs}^{ij} obtained by giving r and s the values 1, 2, 3 form a symmetric 3×3 matrix which exhibits the same behaviour under orthogonal transformations of the x_i coordinates as the matrices $\mathbf{E}$, $\mathbf{A}_r$. Denoting this matrix* by $\mathbf{C}$ and defining the matrices ${}^{(r)}\mathbf{P}$, ${}^{(r)}\mathbf{Q}$ by

$${}^{(r)}\mathbf{P} = \| {}^{(r)}P_{ij} \|, \quad {}^{(r)}\mathbf{Q} = \| {}^{(r)}Q_{ij} \|, \quad (4.7)$$

we may write

$${}^{(r)}P_{1p} C_{pq}^{ij} {}^{(r)}Q_{q1} = [{}^{(r)}\mathbf{P} \mathbf{C} {}^{(r)}\mathbf{Q}]_{11}, \quad (4.8)$$

where the symbol $[]_{11}$ indicates that only the leading element of the matrix product is required. The expression (4.8) is therefore of the same form as those for the transversely isotropic invariants J_r but involves, in addition to the kinematic matrices $\mathbf{E}$, $\mathbf{A}_r$, the transformation matrix $\mathbf{C}$.

The reduction to simpler forms, both of the transversely isotropic coefficients and the invariants J_r thus involves a consideration of the leading term only of a matrix polynomial of symmetric matrices. Since from (4.3)

$$P_{11} = b_{1i} c_{ij} \dots d_{mn} f_{n1} = f_{1i} d_{ij} \dots c_{mn} b_{n1}, \quad (4.9)$$

we may equally well regard P_{11} as the leading term of the symmetric product

$$P_{rs} = \frac{1}{2} [b_{ri} c_{ij} \dots d_{mn} f_{ns} + b_{si} c_{ij} \dots d_{mn} f_{nr}]. \quad (4.10)$$

* Although there are in fact six matrices $\mathbf{C}$ obtained from the six independent choices for i, j , symmetry properties are examined by considering transformations of the x_i coordinates (represented by the suffixes r, s) in the undeformed material alone. The indices i, j preserve the tensor character of τ^{ij} under transformations of the θ^i coordinates. We shall therefore refer to $\mathbf{C}$ as if only one matrix were involved. As an alternative procedure, we could consider the function F and derive the stresses by differentiation after the reduction of F has been carried out. The matrix $\mathbf{Q}$ would then replace $\mathbf{C}$ in the analysis.

5. Some reductions of symmetric matrix products

By applications of the Cayley Hamilton theorem and Peano's theorem, SPENCER & RIVLIN (1959a) have shown that any symmetric isotropic matrix polynomial in r symmetric 3×3 matrices $\mathbf{a}_p$ ($p=1, 2, \dots, r$) may be expressed as a linear combination of the matrix products

$$\mathbf{I}, \quad \mathbf{a}_i, \quad \mathbf{a}_i^2, \quad (\text{I})$$

$$\mathbf{a}_i \mathbf{a}_j + \mathbf{a}_j \mathbf{a}_i, \quad \mathbf{a}_i \mathbf{a}_j^2 + \mathbf{a}_j^2 \mathbf{a}_i, \quad \mathbf{a}_i^2 \mathbf{a}_j^2 + \mathbf{a}_j^2 \mathbf{a}_i^2, \quad (\text{II})$$

$$\mathbf{a}_i \mathbf{a}_j \mathbf{a}_k + \mathbf{a}_k \mathbf{a}_j \mathbf{a}_i, \quad \mathbf{a}_i^2 \mathbf{a}_j \mathbf{a}_k + \mathbf{a}_k \mathbf{a}_j \mathbf{a}_i^2, \quad \mathbf{a}_j \mathbf{a}_i^2 \mathbf{a}_k + \mathbf{a}_k \mathbf{a}_i^2 \mathbf{a}_j, \quad (\text{III})$$

$$\mathbf{a}_i^2 \mathbf{a}_j^2 \mathbf{a}_k + \mathbf{a}_k \mathbf{a}_j^2 \mathbf{a}_i^2, \quad (\text{i}) \quad \mathbf{a}_i \mathbf{a}_j \mathbf{a}_i^2 \mathbf{a}_k + \mathbf{a}_k \mathbf{a}_i^2 \mathbf{a}_j \mathbf{a}_i, \quad (\text{ii}) \quad (\text{IV})$$

$$\mathbf{a}_i \mathbf{a}_j \mathbf{a}_k \mathbf{a}_i^2 + \mathbf{a}_i^2 \mathbf{a}_k \mathbf{a}_j \mathbf{a}_i, \quad (\text{iii})$$

$$\mathbf{a}_i \mathbf{a}_j \mathbf{a}_k \mathbf{a}_l + \mathbf{a}_l \mathbf{a}_k \mathbf{a}_j \mathbf{a}_i, \quad (\text{V})$$

$$\mathbf{a}_i^2 \mathbf{a}_j \mathbf{a}_k \mathbf{a}_l + \mathbf{a}_l \mathbf{a}_k \mathbf{a}_j \mathbf{a}_i^2, \quad \mathbf{a}_i \mathbf{a}_j^2 \mathbf{a}_k \mathbf{a}_l + \mathbf{a}_l \mathbf{a}_k \mathbf{a}_j^2 \mathbf{a}_i, \quad (\text{VI}) \quad (5.1)$$

$$\mathbf{a}_i^2 \mathbf{a}_j^2 \mathbf{a}_k \mathbf{a}_l + \mathbf{a}_l \mathbf{a}_k \mathbf{a}_j^2 \mathbf{a}_i^2, \quad \mathbf{a}_i \mathbf{a}_j^2 \mathbf{a}_k^2 \mathbf{a}_l + \mathbf{a}_l \mathbf{a}_k^2 \mathbf{a}_j^2 \mathbf{a}_i, \quad (\text{VII})$$

$$\mathbf{a}_i \mathbf{a}_j \mathbf{a}_k \mathbf{a}_l \mathbf{a}_i^2 + \mathbf{a}_i^2 \mathbf{a}_l \mathbf{a}_k \mathbf{a}_j \mathbf{a}_i, \quad \mathbf{a}_j \mathbf{a}_i \mathbf{a}_k \mathbf{a}_l \mathbf{a}_i^2 + \mathbf{a}_i^2 \mathbf{a}_l \mathbf{a}_k \mathbf{a}_i \mathbf{a}_j, \quad (\text{VIII})$$

$$\mathbf{a}_j \mathbf{a}_k \mathbf{a}_i \mathbf{a}_l \mathbf{a}_i^2 + \mathbf{a}_i^2 \mathbf{a}_l \mathbf{a}_i \mathbf{a}_k \mathbf{a}_j, \quad \mathbf{a}_j \mathbf{a}_i \mathbf{a}_k \mathbf{a}_i^2 \mathbf{a}_l + \mathbf{a}_l \mathbf{a}_i^2 \mathbf{a}_k \mathbf{a}_i \mathbf{a}_j, \quad (\text{IX})$$

$$\mathbf{a}_i \mathbf{a}_j \mathbf{a}_k \mathbf{a}_l \mathbf{a}_m + \mathbf{a}_m \mathbf{a}_l \mathbf{a}_k \mathbf{a}_j \mathbf{a}_i, \quad (\text{IX})$$

$$\mathbf{a}_i \mathbf{a}_j \mathbf{a}_k \mathbf{a}_l \mathbf{a}_m^2 + \mathbf{a}_m^2 \mathbf{a}_l \mathbf{a}_k \mathbf{a}_j \mathbf{a}_i, \quad \mathbf{a}_j \mathbf{a}_k \mathbf{a}_l \mathbf{a}_m^2 \mathbf{a}_i + \mathbf{a}_i \mathbf{a}_m^2 \mathbf{a}_l \mathbf{a}_k \mathbf{a}_j, \quad (\text{X})$$

$$\mathbf{a}_k \mathbf{a}_l \mathbf{a}_m^2 \mathbf{a}_i \mathbf{a}_j + \mathbf{a}_j \mathbf{a}_i \mathbf{a}_m^2 \mathbf{a}_l \mathbf{a}_k, \quad (\text{X})$$

$$\mathbf{a}_i \mathbf{a}_j \mathbf{a}_k \mathbf{a}_l \mathbf{a}_m \mathbf{a}_n + \mathbf{a}_n \mathbf{a}_m \mathbf{a}_l \mathbf{a}_k \mathbf{a}_j \mathbf{a}_i. \quad (\text{XI})$$

In the table (5.1) all possible selections of the suffixes i, j, k, l, m, n (all different) are chosen from the numbers $1, 2, \dots, r$; the coefficients in the resulting matrix, polynomial are polynomial scalar invariants under the full orthogonal group that is, polynomials in the first kind of invariant I_r in (4.4).

For transversely isotropic bodies, the additional invariants J_r are all included in the set obtained by taking the leading terms of the products (I) to (XI) and substituting the matrices $\mathbf{E}, \mathbf{A}_p$ ($p=1, 2, \dots, n$) for $\mathbf{a}_1, \mathbf{a}_2, \dots, \mathbf{a}_{n+1}$. The coefficients (4.8) are obtained similarly from the corresponding set for $n+2$ matrices. In this case we substitute $\mathbf{C}, \mathbf{E}, \mathbf{A}_p$ ($p=1, 2, \dots, n$) for $\mathbf{a}_1, \mathbf{a}_2, \dots, \mathbf{a}_{n+2}$ and choose only those invariants in which $\mathbf{C}$ ($=\mathbf{a}_1$) occurs linearly.

By further analysis based on the Cayley Hamilton theorem the writer (1958) has shown that the products (ii) and (iii) of set (IV) can be expressed in terms of each other, together with products of lower degree in the matrices $\mathbf{a}_i$; either of the types (ii) or (iii) may therefore be omitted from the scheme (5.1) without loss of generality. Furthermore, from a systematic examination of the products (VII), (VIII), (X) and (XI) carried out by SPENCER & RIVLIN* we may infer that these may also be expressed in terms of products of lower degree and may thus be omitted from (5.1). In the resulting scheme, the symmetry of the system is

* *Private communication.* I am indebted to Professor A. E. GREEN for drawing my attention to this work which, I understand, is to be published shortly.

preserved at the expense of still further redundancies in the coefficients. For example, from any four matrices $\mathbf{a}_i, \mathbf{a}_j, \mathbf{a}_k, \mathbf{a}_l$ we may derive twelve symmetrical products of type (V) by taking all possible permutations of the letters i, j, k, l . Not all of these products are linearly independent, however, for relations between them follow as a consequence of the Cayley Hamilton theorem. We may observe that any reduction in the number of coefficients (5.1) leads to a corresponding reduction in the number of invariants I_r and J_r .

To achieve these further reductions we require the Cayley Hamilton theorem

$$\mathbf{a}^3 - \mathbf{a}^2 \operatorname{tr} \mathbf{a} + \frac{1}{2} \mathbf{a} [(\operatorname{tr} \mathbf{a})^2 - \operatorname{tr} \mathbf{a}^2] - \frac{1}{6} \mathbf{I} [(\operatorname{tr} \mathbf{a})^3 - 3 \operatorname{tr} \mathbf{a} \operatorname{tr} \mathbf{a}^2 + 2 \operatorname{tr} \mathbf{a}^3] = 0 \quad (5.2)$$

$(\operatorname{tr} \mathbf{a} = \text{trace } \mathbf{a}),$

for a 3×3 matrix $\mathbf{a}$, and the generalization

$$\begin{aligned} & \mathbf{a} \mathbf{b} \mathbf{c} + \mathbf{a} \mathbf{c} \mathbf{b} + \mathbf{b} \mathbf{a} \mathbf{c} + \mathbf{b} \mathbf{c} \mathbf{a} + \mathbf{c} \mathbf{a} \mathbf{b} + \mathbf{c} \mathbf{b} \mathbf{a} \\ &= (\mathbf{b} \mathbf{c} + \mathbf{c} \mathbf{b}) \operatorname{tr} \mathbf{a} + (\mathbf{a} \mathbf{c} + \mathbf{c} \mathbf{a}) \operatorname{tr} \mathbf{b} + (\mathbf{a} \mathbf{b} + \mathbf{b} \mathbf{a}) \operatorname{tr} \mathbf{c} + \\ & \quad + \mathbf{a} (\operatorname{tr} \mathbf{b} \mathbf{c} - \operatorname{tr} \mathbf{b} \operatorname{tr} \mathbf{c}) + \mathbf{b} (\operatorname{tr} \mathbf{c} \mathbf{a} - \operatorname{tr} \mathbf{c} \operatorname{tr} \mathbf{a}) + \\ & \quad + \mathbf{c} (\operatorname{tr} \mathbf{a} \mathbf{b} - \operatorname{tr} \mathbf{a} \operatorname{tr} \mathbf{b}) + \\ & \quad + \mathbf{I} (\operatorname{tr} \mathbf{a} \operatorname{tr} \mathbf{b} \operatorname{tr} \mathbf{c} - \operatorname{tr} \mathbf{a} \operatorname{tr} \mathbf{b} \mathbf{c} - \operatorname{tr} \mathbf{b} \operatorname{tr} \mathbf{c} \mathbf{a} - \\ & \quad - \operatorname{tr} \mathbf{c} \operatorname{tr} \mathbf{a} \mathbf{b} + \operatorname{tr} \mathbf{a} \mathbf{b} \mathbf{c} + \operatorname{tr} \mathbf{c} \mathbf{b} \mathbf{a}), \end{aligned} \quad (5.3)$$

for three matrices $\mathbf{a}, \mathbf{b}, \mathbf{c}$ obtained by RIVLIN (1955) by substituting $\mathbf{a} + \mathbf{b}$ and $\mathbf{a} + \mathbf{b} + \mathbf{c}$ for $\mathbf{a}$ in (5.2).

In each of the equations (5.2), (5.3) the left hand member is a matrix polynomial of the third degree in $\mathbf{a}, \mathbf{b}, \mathbf{c}$ while all of the terms on the right hand side are of lower degree, the coefficients of these lower degree terms being scalar polynomial invariants. To abbreviate the subsequent analysis we shall use the following convention. If a matrix polynomial $\mathbf{P}$ which is homogeneous and of the r^{th} degree in matrices $\mathbf{a}, \mathbf{b}, \mathbf{c}, \dots$ can be expressed as a polynomial in which all of the terms are of degree $\leq r-1$ we shall describe $\mathbf{P}$ as reducible and write

$$\mathbf{P} \equiv 0.$$

With this convention, (5.2) and (5.3) may be re-written

$$\mathbf{a}^3 \equiv 0, \quad (5.2')$$

$$\mathbf{a} \mathbf{b} \mathbf{c} + \mathbf{a} \mathbf{c} \mathbf{b} + \mathbf{b} \mathbf{a} \mathbf{c} + \mathbf{b} \mathbf{c} \mathbf{a} + \mathbf{c} \mathbf{a} \mathbf{b} + \mathbf{c} \mathbf{b} \mathbf{a} \equiv 0. \quad (5.3')$$

If $\mathbf{P}$ and $\mathbf{Q}$ are two matrix polynomials, homogeneous and of degree r , for which

$$k_1 \mathbf{P} + k_2 \mathbf{Q} \equiv 0, \quad (5.4)$$

k_1 and k_2 being numerical coefficients, then obviously we may write

$$k_1 \mathbf{P} \equiv -k_2 \mathbf{Q}. \quad (5.5)$$

In general, if $\mathbf{P}_1, \mathbf{P}_2, \dots, \mathbf{P}_n$ are a system of n matrix polynomials which are homogeneous and of degree r , and

$$\alpha_{1i} \mathbf{P}_1 + \alpha_{2i} \mathbf{P}_2 + \dots + \alpha_{ni} \mathbf{P}_n \equiv 0 \quad (i = 1, 2, \dots, m; m < n),$$

are m linear relations between them with constant coefficients α_{pq} , then m of the matrices $\mathbf{P}_s$ can be expressed linearly in terms of the remainder together with terms of degree $\leq r-1$ by means of the ordinary processes for solving simultaneous equations, provided the coefficient matrix $\|\alpha_{pq}\|$ has rank m .

In dealing with symmetric products we use the notation

$$(abc \dots def) = abc \dots def + fed \dots cba, \quad (5.6)$$

for symmetric matrix polynomials and write

$$[abc \dots def], \quad (5.7)$$

for the sum of all terms formed from the product $abc \dots def$ by a full permutation of the letters $a, b, c, \dots, d, e, f$. With this notation, (5.3) or (5.3') may be written as

$$(abc) + (acb) + (bac) \equiv 0,$$

or as

$$[abc] \equiv 0. \quad (5.8)$$

We define the *symmetric* multiplication of the symmetric product $abc \dots def$ by the product $tu \dots vw$ as

$$\begin{aligned} & tu \dots vw(abc \dots def) + (abc \dots def)vw \dots ut \\ &= (tu \dots vwabc \dots def) + (abc \dots defvw \dots ut), \end{aligned} \quad (5.9)$$

and observe that a symmetric polynomial results from this operation. We define the *symmetric* substitution of d by $tu \dots vw$ in the symmetric product $(ab \dots cde \dots fh)$ by

$$\begin{aligned} & (ab \dots c(tu \dots vw)e \dots fh) \\ &= ab \dots ctu \dots vwe \dots fh + hf \dots etu \dots vwc \dots ba + \\ &+ ab \dots cww \dots ute \dots fh + hf \dots ewv \dots utc \dots ba \\ &= (ab \dots ctu \dots vwe \dots fh) + (ab \dots cww \dots ute \dots fh), \end{aligned} \quad (5.10)$$

the result again being a symmetric polynomial.

From any four different matrices a, b, c, d we may form the twelve symmetric products

$$\begin{aligned} & (abcd), (abdc), (acbd), (acdb), (adb c), (bacd), \quad (i) \\ & (adcb), (badc), (bcad), (bdac), (cabd), (cbad), \quad (ii) \end{aligned} \quad (5.11)$$

of type (V). Between these, we may obtain by the symmetric substitution of a by ad in (5.3') or (5.8) the relation

$$(adb c) + (adcb) + (badc) + (cadb) + (bcad) + (cbad) \equiv 0,$$

or

$$[(ad)bc] \equiv 0. \quad (5.12)$$

By similar substitutions of this type, as indicated, we may derive the five further relations

$$\begin{aligned} & [(ab)cd] \equiv 0, \quad [(ac)bd] \equiv 0, \\ & [(bc)ad] \equiv 0, \quad [(bd)ac] \equiv 0, \\ & [(cd)ab] \equiv 0. \end{aligned} \quad (5.13)$$

These six relations are linearly independent in the products (5.11) and yield the solutions

$$\begin{aligned}
 (\mathbf{a} \mathbf{d} \mathbf{c} \mathbf{b}) &\equiv -\{(\mathbf{a} \mathbf{b} \mathbf{c} \mathbf{d}) + (\mathbf{a} \mathbf{b} \mathbf{d} \mathbf{c}) + (\mathbf{a} \mathbf{c} \mathbf{b} \mathbf{d}) + (\mathbf{a} \mathbf{c} \mathbf{d} \mathbf{b}) + (\mathbf{a} \mathbf{d} \mathbf{b} \mathbf{c})\}, \\
 (\mathbf{b} \mathbf{a} \mathbf{d} \mathbf{c}) &\equiv (\mathbf{a} \mathbf{c} \mathbf{b} \mathbf{d}) + (\mathbf{a} \mathbf{d} \mathbf{b} \mathbf{c}) - (\mathbf{b} \mathbf{a} \mathbf{c} \mathbf{d}), \\
 (\mathbf{b} \mathbf{c} \mathbf{a} \mathbf{d}) &\equiv (\mathbf{a} \mathbf{b} \mathbf{d} \mathbf{c}) + (\mathbf{a} \mathbf{d} \mathbf{b} \mathbf{c}) - (\mathbf{b} \mathbf{a} \mathbf{c} \mathbf{d}), \\
 (\mathbf{b} \mathbf{d} \mathbf{a} \mathbf{c}) &\equiv (\mathbf{b} \mathbf{a} \mathbf{c} \mathbf{d}) + (\mathbf{a} \mathbf{b} \mathbf{c} \mathbf{d}) - (\mathbf{a} \mathbf{d} \mathbf{b} \mathbf{c}), \\
 (\mathbf{c} \mathbf{a} \mathbf{b} \mathbf{d}) &\equiv -\{(\mathbf{a} \mathbf{b} \mathbf{c} \mathbf{d}) + (\mathbf{a} \mathbf{b} \mathbf{d} \mathbf{c}) + (\mathbf{a} \mathbf{c} \mathbf{b} \mathbf{d}) + (\mathbf{a} \mathbf{c} \mathbf{d} \mathbf{b}) + (\mathbf{b} \mathbf{a} \mathbf{c} \mathbf{d})\}, \\
 (\mathbf{c} \mathbf{b} \mathbf{a} \mathbf{d}) &\equiv (\mathbf{b} \mathbf{a} \mathbf{c} \mathbf{d}) + (\mathbf{a} \mathbf{c} \mathbf{d} \mathbf{b}) - (\mathbf{a} \mathbf{d} \mathbf{b} \mathbf{c}).
 \end{aligned} \tag{5.14}$$

Each symmetric product of the set (ii) in (5.11) can therefore be expressed as a linear combination of the products (i), together with terms of lower degree. We observe that in (5.14) the formula for $(\mathbf{a} \mathbf{d} \mathbf{c} \mathbf{b})$ or for $(\mathbf{c} \mathbf{a} \mathbf{b} \mathbf{d})$ may be replaced by one of the relations

$$\begin{aligned}
 (\mathbf{a} \mathbf{d} \mathbf{c} \mathbf{b}) - (\mathbf{c} \mathbf{a} \mathbf{b} \mathbf{d}) &= (\mathbf{b} \mathbf{a} \mathbf{c} \mathbf{d}) - (\mathbf{a} \mathbf{d} \mathbf{b} \mathbf{c}) \\
 &= (\mathbf{a} \mathbf{c} \mathbf{b} \mathbf{d}) - (\mathbf{b} \mathbf{a} \mathbf{d} \mathbf{c}) \\
 &= (\mathbf{a} \mathbf{b} \mathbf{d} \mathbf{c}) - (\mathbf{b} \mathbf{c} \mathbf{a} \mathbf{d}) \\
 &= (\mathbf{b} \mathbf{d} \mathbf{a} \mathbf{c}) - (\mathbf{a} \mathbf{b} \mathbf{c} \mathbf{d}) \\
 &= (\mathbf{c} \mathbf{b} \mathbf{a} \mathbf{d}) - (\mathbf{a} \mathbf{c} \mathbf{d} \mathbf{b}),
 \end{aligned} \tag{5.15}$$

connecting $(\mathbf{a} \mathbf{d} \mathbf{c} \mathbf{b})$ and $(\mathbf{c} \mathbf{a} \mathbf{b} \mathbf{d})$. No further reductions result from the symmetric multiplication of (5.8) by the remaining matrix $\mathbf{d}$, for it can be shown without difficulty that

$$\begin{aligned}
 3(\mathbf{d}[\mathbf{a} \mathbf{b} \mathbf{c}]) &= 3\mathbf{d}[\mathbf{a} \mathbf{b} \mathbf{c}] + 3[\mathbf{a} \mathbf{b} \mathbf{c}]\mathbf{d} \\
 &= 2[(\mathbf{b} \mathbf{c})\mathbf{a} \mathbf{d}] + 2[(\mathbf{a} \mathbf{c})\mathbf{b} \mathbf{d}] + 2[(\mathbf{a} \mathbf{b})\mathbf{c} \mathbf{d}] - \\
 &\quad - [(\mathbf{a} \mathbf{d})\mathbf{b} \mathbf{c}] - [(\mathbf{b} \mathbf{d})\mathbf{a} \mathbf{c}] - [(\mathbf{c} \mathbf{d})\mathbf{a} \mathbf{b}],
 \end{aligned} \tag{5.16}$$

with corresponding formulae for $(\mathbf{b}[\mathbf{a} \mathbf{c} \mathbf{d}])$, $(\mathbf{c}[\mathbf{a} \mathbf{b} \mathbf{d}])$ and $(\mathbf{a}[\mathbf{b} \mathbf{c} \mathbf{d}])$. For any given choice of the suffixes i, j, k, l from the numbers 1 to r we may therefore obtain the set of products

$$\begin{aligned}
 (\mathbf{a}_i \mathbf{a}_j \mathbf{a}_k \mathbf{a}_l), \quad (\mathbf{a}_i \mathbf{a}_j \mathbf{a}_l \mathbf{a}_k), \quad (\mathbf{a}_i \mathbf{a}_k \mathbf{a}_j \mathbf{a}_l), \\
 (\mathbf{a}_i \mathbf{a}_k \mathbf{a}_l \mathbf{a}_j), \quad (\mathbf{a}_i \mathbf{a}_l \mathbf{a}_j \mathbf{a}_k), \quad (\mathbf{a}_j \mathbf{a}_i \mathbf{a}_k \mathbf{a}_l),
 \end{aligned} \tag{5.17}$$

of type (V). An equivalent result may be obtained by differentiation from the integrity basis for invariants involving five matrices derived by SPENCER & RIVLIN (1959b).

Analogous reductions may be carried out for the remaining higher degree products of types (VI) and (IX), and a large number of redundant terms eliminated in this way. It is, in fact, just possible that products of type (IX) may prove to be completely reducible. For if we denote the system (5.14) by $\{\mathbf{a} \mathbf{b} \mathbf{c} \mathbf{d}\}$ and the sets of six equations derived by the symmetric substitutions of $\mathbf{a}$ by $\mathbf{a} \mathbf{e}$,

$\mathbf{b}$ by $\mathbf{b}e, \dots$ by $\{(\mathbf{a}e)\mathbf{b}cd\}, \{\mathbf{a}(\mathbf{b}e)\mathbf{c}d\}, \dots$ the systems

$$\begin{aligned} & \{(\mathbf{a}e)\mathbf{b}cd\}, \{\mathbf{a}(\mathbf{b}e)\mathbf{c}d\}, \{\mathbf{a}b(\mathbf{c}e)d\}, \{\mathbf{a}b\mathbf{c}(\mathbf{d}e)\}, \\ & \{(\mathbf{a}d)\mathbf{b}ce\}, \{\mathbf{a}(\mathbf{b}d)\mathbf{c}e\}, \{\mathbf{a}b(\mathbf{c}d)e\}, \\ & \{(\mathbf{a}e)\mathbf{b}de\}, \{\mathbf{a}(\mathbf{b}c)\mathbf{d}e\}, \{(\mathbf{a}b)\mathbf{c}de\}, \end{aligned}$$

yield exactly sixty relations between the sixty symmetric products derived from $(\mathbf{a}b\mathbf{c}d\mathbf{e})$ by permutations of the letters $\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d}, \mathbf{e}$. If these relations are linearly independent, each of the products of type (IX) can be expressed entirely in terms of products of lower degree. We shall not, however, examine these further reductions here, since for the transversely isotropic case all of the invariants I_r, J_r formed from matrix products of higher degree than the fourth by taking the traces and the leading terms respectively can be reduced completely by the method of § 8.

6. Transversely isotropic bodies: an alternative formulation

An alternative description of the invariant system for transversely isotropic bodies may be obtained from (3.17). We again consider the system of r symmetric 3×3 matrices $\mathbf{a}_1, \mathbf{a}_2, \dots, \mathbf{a}_r$ but in this section we shall denote by $\mathbf{b}_k$ the 2×2 matrix derived from $\mathbf{a}_k$ by omitting the first row and column, and from the remaining elements of the matrices $\mathbf{a}_k$ construct an additional system of 2×2 matrices. Denoting any two different matrices $\mathbf{a}_i, \mathbf{a}_k$ of the set $\mathbf{a}_p$ ($p = 1, 2, \dots, r$) for which $i < k$, by $\mathbf{a}, \mathbf{b}$ respectively, so that

$$\mathbf{a} = \|a_{ij}\|, \quad \mathbf{b} = \|b_{ij}\|, \quad (6.1)$$

we write

$$\mathbf{b}_i = \|a_{\alpha\beta}\|, \quad \mathbf{c}_i = \|a_{1\alpha}a_{\beta 1}\|, \quad \mathbf{c}_{ik} = \|a_{1\alpha}b_{\beta 1}\| \quad (i < k, a_{1\alpha} \neq b_{1\alpha}, \alpha, \beta = 2, 3). \quad (6.2)$$

This gives a system of $2r$ symmetric 2×2 matrices $\mathbf{b}_i, \mathbf{c}_i$ and $\frac{1}{2}r(r-1)$ unsymmetric 2×2 matrices $\mathbf{c}_{ik}$. From (3.17) it follows that any of the isotropic or transversely isotropic invariants for the system of 3×3 matrices $\mathbf{a}_k$ may be expressed in terms of the isotropic invariants for the system of 2×2 matrices $\mathbf{b}_i, \mathbf{c}_i, \mathbf{c}_{ik}$ together with the leading terms (e.g. a_{11}) of each of the matrices $\mathbf{a}_k$. Furthermore, each of these invariants of the system (6.2) can contain at most one of the matrices $\mathbf{c}_i, \mathbf{c}_{ik}$ linearly.

To reduce the system of invariants derived from (6.2) we make use of the following results derived by RIVLIN (1955) by applications of the Cayley Hamilton theorem:

(i) Any matrix polynomial $\mathbf{P}$ in n symmetric 2×2 matrices $\mathbf{B}_p$ ($p = 1, 2, \dots, n$) can be expressed in the form

$$\mathbf{P} = \varphi_0 \mathbf{I} + \sum_{i=1}^n \varphi_i \mathbf{B}_i + \sum_{i=1}^n \sum_{k=1}^n \varphi_{ik} \mathbf{B}_i \mathbf{B}_k \quad (i \neq k), \quad (6.3)$$

where φ_0, φ_i and φ_{ik} are polynomials in the invariants formed from the matrices $\mathbf{B}_p$ and $\mathbf{I}$ is the unit matrix.

(ii) Any polynomial invariant formed from the elements of the system of 2×2 matrices $\mathbf{B}_p$ can be expressed as a polynomial in the quantities

$$\text{tr } \mathbf{B}_i, \quad \text{tr } \mathbf{B}_i \mathbf{B}_k.$$

From (3.17) we observe that any polynomial invariant of the matrices (6.2) is expressible as a polynomial in the traces of these matrices. It follows immediately from (ii) that all invariants formed from the symmetric matrices $\mathbf{b}_i$, $\mathbf{c}_i$ are expressible as polynomials in the quantities

$$\text{tr } \mathbf{b}_i, \quad \text{tr } \mathbf{b}_i \mathbf{b}_k, \quad \text{tr } \mathbf{c}_i, \quad \text{tr } \mathbf{b}_i \mathbf{c}_k, \quad (6.4)$$

if we remember that all invariants must be linear in $\mathbf{c}_i$. Again, any invariant containing elements of $\mathbf{c}_{i_k}$ must be expressible in the form $\text{tr } \mathbf{c}_{i_k} \mathbf{P}$ where $\mathbf{P}$ is a matrix polynomial containing only the matrices $\mathbf{b}_k$. With this result, the expansion (6.3) implies that any polynomial invariant containing the matrices $\mathbf{c}_{i_k}$ is expressible as a polynomial in

$$\text{tr } \mathbf{c}_{i_k}, \quad \text{tr } \mathbf{c}_{i_k} \mathbf{b}_l, \quad \text{tr } \mathbf{c}_{i_k} \mathbf{b}_l \mathbf{b}_m, \quad \text{tr } \mathbf{b}_i, \quad \text{tr } \mathbf{b}_i \mathbf{b}_j \quad (l \neq m). \quad (6.5)$$

Remembering (6.4) and (6.5) we deduce that the invariants for the transversely isotropic case can be expressed as a polynomial in the quantities

$$a_{11}, \quad a_{\alpha\alpha}, \quad a_{\alpha\beta} b_{\beta\alpha}, \quad a_{1\alpha} b_{\alpha 1}, \quad a_{1\alpha} c_{\alpha\beta} b_{\beta 1}, \quad a_{1\alpha} c_{\alpha\beta} d_{\beta\gamma} b_{\gamma 1}, \quad (6.6)$$

where $a_{\alpha\beta}$, $b_{\alpha\beta}$, $c_{\alpha\beta}$, $d_{\alpha\beta}$ represent elements of any four of the matrices $\mathbf{a}_i$ subject only to the condition that in the last of (6.6) $c_{\alpha\beta}$, $d_{\alpha\beta}$ are elements of different matrices and likewise the matrices $\mathbf{a}_{ij}$, $\mathbf{b}_{ij}$ are not identical. If $c_{\alpha\beta} = d_{\alpha\beta}$ the last of the invariants (6.6) can be expressed as a polynomial in the simpler invariants by using the Cayley Hamilton theorem for a 2×2 matrix. The system (6.6) represents a different expression of the total set I_r , J_r of the invariants (4.4) and not merely of the invariants J_r .

7. Invariant relations for two matrices

The connexion between the formulation of § 6 and that of the preceding sections may be illustrated by examining the systems of invariants formed from the two symmetric 3×3 matrices

$$\mathbf{a} = \|a_{ij}\|, \quad \mathbf{b} = \|b_{ij}\|. \quad (7.1)$$

Corresponding to the theory of §§ 3 to 5 we have the system

$$\begin{aligned} I_1 &= a_{ii}, & I_2 &= a_{ij} a_{ji}, & I_3 &= a_{ij} a_{jk} a_{ki}, & I_4 &= a_{11}, & I_5 &= a_{1i} a_{i1}, \\ I'_1 &= b_{ii}, & I'_2 &= b_{ij} b_{ji}, & I'_3 &= b_{ij} b_{jk} b_{ki}, & I'_4 &= b_{11}, & I'_5 &= b_{1i} b_{i1}, \\ J_1 &= a_{ij} b_{ji}, & J_2 &= a_{ij} a_{jk} b_{ki}, & J_3 &= a_{ij} b_{jk} b_{ki}, & J_4 &= a_{ij} a_{jk} b_{kl} b_{li}, \\ J_5 &= a_{1i} b_{i1}, & J_6 &= a_{1i} a_{ij} b_{j1}, & J_7 &= a_{1i} b_{ij} b_{j1}, & J_8 &= a_{1i} a_{ij} b_{jk} b_{ki}. \end{aligned} \quad (7.2)$$

Of these, the invariants $I_1, I_2, I_3, I'_1, I'_2, I'_3, J_1, J_2, J_3, J_4$ are those appropriate to isotropic bodies and have been given by RIVLIN (1955). The remainder apply only to transversely isotropic materials. From (6.6) we obtain the alternative set of invariants

$$\begin{aligned} K_1 &= a_{\alpha\alpha}, & K_2 &= a_{\alpha\beta} a_{\beta\alpha}, & K_3 &= a_{11}, & K_4 &= a_{1\alpha} a_{\alpha 1}, & K_5 &= a_{1\alpha} a_{\alpha\beta} a_{\beta 1}, \\ K'_1 &= b_{\alpha\alpha}, & K'_2 &= b_{\alpha\beta} b_{\beta\alpha}, & K'_3 &= b_{11}, & K'_4 &= b_{1\alpha} b_{\alpha 1}, & K'_5 &= b_{1\alpha} b_{\alpha\beta} b_{\beta 1}, \\ L_1 &= a_{\alpha\beta} b_{\beta\alpha}, & L_2 &= a_{1\alpha} b_{\alpha 1}, & L_3 &= a_{1\alpha} a_{\alpha\beta} b_{\beta 1}, & L_4 &= a_{1\alpha} a_{\alpha\beta} b_{\beta 1}, \\ L_5 &= a_{1\alpha} b_{\alpha\beta} a_{\beta 1}, & L_6 &= b_{1\alpha} a_{\alpha\beta} b_{\beta 1}, & L_7 &= a_{1\alpha} a_{\alpha\beta} b_{\beta\gamma} b_{\gamma 1}, & L_8 &= a_{1\alpha} b_{\alpha\beta} a_{\beta\gamma} b_{\gamma 1}. \end{aligned} \quad (7.3)$$

To express the system (7.2) in terms of the invariants (7.3) we note that for any 2×2 matrix $\mathbf{c}$ we have the Cayley Hamilton theorem

$$\mathbf{c}^2 = \mathbf{c} \operatorname{tr} \mathbf{c} + \frac{1}{2} \mathbf{I} [\operatorname{tr} \mathbf{c}^2 - (\operatorname{tr} \mathbf{c})^2], \quad (7.4)$$

and by post multiplying by any other 2×2 matrix $\mathbf{d}$ and taking the trace of the resulting equation we deduce that

$$\operatorname{tr} \mathbf{c}^2 \mathbf{d} = \operatorname{tr} \mathbf{c} \operatorname{tr} \mathbf{c} \mathbf{d} + \frac{1}{2} \operatorname{tr} \mathbf{d} [\operatorname{tr} \mathbf{c}^2 - (\operatorname{tr} \mathbf{c})^2]. \quad (7.5)$$

Again, if we multiply corresponding sides of (7.4) by the analogous relation for $\mathbf{d}$ and take the trace of the resulting expression we have

$$\operatorname{tr} \mathbf{c}^2 \mathbf{d}^2 = \frac{1}{2} \operatorname{tr} \mathbf{c}^2 \operatorname{tr} \mathbf{d}^2 + \operatorname{tr} \mathbf{c} \mathbf{d} \operatorname{tr} \mathbf{c} \operatorname{tr} \mathbf{d} - \frac{1}{2} (\operatorname{tr} \mathbf{c})^2 (\operatorname{tr} \mathbf{d})^2. \quad (7.6)$$

Relations between the invariants (7.2) and (7.3) are now readily obtained by translating these into the symbolic notation of § 3, employing the first of (3.12) and using (7.4) to (7.6) where necessary to simplify the expressions obtained after regrouping the symbols $a_i, a'_i, a''_i, \dots, b_i, b'_i, b''_i, \dots$ into the elements a_{ij}, b_{ij} . This process yields, in the case of J_2 , for example

$$\begin{aligned} J_2 &= a_{ij} a_{jk} b_{ki} = (a a') (a' b) (b a) \\ &= [(a|a') + (l a) (l a')] [(a'|b) + (l a') (l b)] [(b|a) + (l b) (l a)] \\ &= a_{\alpha\beta} a_{\beta\gamma} b_{\gamma\alpha} + a_{1\alpha} b_{\alpha\beta} a_{\beta 1} + 2 a_{1\alpha} a_{\alpha\beta} b_{\beta 1} + \\ &\quad + 2 a_{11} a_{1\alpha} b_{\alpha 1} + b_{11} a_{1\alpha} a_{\alpha 1} + a_{11}^2 b_{11}. \end{aligned}$$

The first of these terms can be simplified by making use of (7.5) with $\mathbf{a}, \mathbf{b}$ for $\mathbf{c}, \mathbf{d}$ respectively; the remainder can be recognised as invariants of the set (7.3). By means of this procedure we obtain the relations

$$\begin{aligned} I_1 &= K_1 + K_3, \\ I_2 &= K_2 + 2K_4 + K_3^2, \\ I_3 &= \frac{3}{2} K_1 K_2 - \frac{1}{2} K_1^3 + 3K_5 + 3K_3 K_4 + K_3^3, \\ I_4 &= K_3, \\ I_5 &= K_4 + K_3^2, \\ J_1 &= L_1 + 2L_2 + K_3 K'_3, \\ J_2 &= 2L_3 + L_5 + K_1 L_1 + 2K_3 L_2 + \frac{1}{2} K'_1 (K_2 - K_1^2) + K'_3 (K_4 + K_3^2), \\ J_3 &= 2L_4 + L_6 + K'_1 L_1 + 2K'_3 L_2 + \frac{1}{2} K_1 (K'_2 - K_1'^2) + K_3 (K'_4 + K_3'^2), \\ J_4 &= 2L_7 + L_2^2 + 2(K_3 L_4 + K'_3 L_3) + 2K_3 K'_3 L_2 + K_1 K'_1 L_1 + \\ &\quad + K_1 L_6 + K'_1 L_5 + (K_4 + K_3^2) (K'_4 + K_3'^2) + \frac{1}{2} (K_2 K'_2 - K_1^2 K_1'^2) + \\ &\quad + \frac{1}{2} K_4 (K'_2 - K_1'^2) + \frac{1}{2} K'_4 (K_2 - K_1^2), \\ J_5 &= L_2 + K_3 K'_3, \\ J_6 &= L_3 + K_3 L_2 + K'_3 (K_4 + K_3^2), \\ J_7 &= L_4 + K'_3 L_2 + K_3 (K'_4 + K_3'^2), \\ J_8 &= L_7 + K_3 L_4 + K'_3 L_3 + K_3 K'_3 L_2 + (K_4 + K_3^2) (K'_4 + K_3'^2), \end{aligned} \quad (7.7)$$

the invariant L_8 evidently being redundant.

The converse process yields the formulae

$$\begin{aligned}
 K_1 &= I_1 - I_4, \\
 K_2 &= I_2 - 2I_5 + I_4^2, \\
 K_3 &= I_4, \\
 K_4 &= I_5 - I_4^2, \\
 K_5 &= \frac{1}{2} I_4 (I_2 - I_1^2 + 2I_4^2) + I_5 (I_1 - 2I_4) + \frac{1}{6} (I_1^3 - 3I_1 I_2 + 2I_3), \\
 L_1 &= J_1 - 2J_5 + I_4 I_4', \\
 L_2 &= J_5 - I_4 I_4', \\
 L_3 &= J_6 - I_4 J_5 - I_4' (I_5 - I_4^2), \\
 L_4 &= J_7 - I_4' J_5 - I_4 (I_5' - I_4'^2), \\
 L_5 &= J_2 - 2J_6 - (I_1 - I_4) (J_1 - 2J_5) + \\
 &\quad + \frac{1}{2} (I_1' - I_4') (I_1^2 - I_2) + I_1' (I_5 - I_1 I_4) + I_4^2 I_4', \\
 L_6 &= J_3 - 2J_7 - (I_1' - I_4') (J_1 - 2J_5) + \\
 &\quad + \frac{1}{2} (I_1 - I_4) (I_1'^2 - I_2') + I_1 (I_5' - I_1' I_4') + I_4 I_4'^2, \\
 L_7 &= J_8 - (I_4 J_7 + I_4' J_6) + I_4 I_4' J_5 - (I_5 - I_4^2) (I_5' - I_4'^2).
 \end{aligned} \tag{7.8}$$

In this case, to simplify the formulae resulting from the rearrangement of the symbols, it is necessary to make use of the Cayley Hamilton theorem (5.2), the formula obtained by putting $\mathbf{c} = \mathbf{a}$ in (5.3), and the relation derived from this by interchange of $\mathbf{a}$ and $\mathbf{b}$. Corresponding to the relations between I_r and K_r in (7.7) and (7.8) there exist analogous relations between I_r' and K_r' .

By eliminating L_7 from the expressions for J_4 and J_8 and substituting for K_r, K_r', L_s ($r=1$ to $5, s=1$ to 6) from (7.8), either J_4 or J_8 may be expressed as a polynomial in the remaining invariants (7.7). A straight-forward calculation yields

$$\begin{aligned}
 J_4 - 2J_8 &= J_5^2 + (2J_5 - J_1) (I_1 - I_4) (I_1' - I_4') - 2J_5 I_4 I_4' + \\
 &\quad + (J_2 - 2J_6) (I_1' - I_4') + (J_3 - 2J_7) (I_1 - I_4) + \\
 &\quad + \frac{1}{2} (I_2 - I_5) (I_2' - I_5') - \frac{3}{2} I_5 I_5' - \\
 &\quad - \frac{1}{2} [I_2 (I_1' - I_4')^2 + I_2' (I_1 - I_4)^2] + \\
 &\quad + \frac{1}{2} (I_1^2 I_5' + I_1' I_5^2) + \\
 &\quad + \frac{1}{2} (I_1 I_1' - I_1 I_4' - I_1' I_4)^2.
 \end{aligned} \tag{7.9}$$

8. A further reduction for transversely isotropic bodies

By the procedure of the preceding section we may effect a considerable reduction in the number of coefficients and invariants of the type discussed in § 5 which are necessary for the general representation of a transversely isotropic body. The steps in this reduction may be summarized as follows:

(i) A given invariant I of the system of 3×3 matrices $\mathbf{a}_k$ having the form (3.16) is expressed, if possible, as a polynomial in simpler invariants by means of reductions based on the Cayley Hamilton theorem and Peano's theorem; an integrity basis for the isotropic invariants derived by this method is given by SPENCER & RIVLIN (1959a, b) and the additional invariants required for transversely isotropic bodies are given in § 5.

(ii) Each invariant of the 3×3 system thus obtained is expressed by means of the symbolic method of § 7 as a polynomial in the invariants of the system (6.2) together with the leading terms of the constituent matrices, these leading terms being themselves invariant under the transversely isotropic group of transformations.

(iii) The invariants of the system of 2×2 matrices (6.2) are expressed by means of the Cayley Hamilton theorem as polynomials in the invariants (6.6).

(iv) By a reversal of step (ii) again using symbolic methods, each invariant of the type (6.6) is replaced by a polynomial in the isotropic and transversely isotropic invariants of the system of 3×3 matrices $\mathbf{a}_k$.

(v) The resulting invariant system is simplified further, if possible, by applications of the Cayley Hamilton theorem and its generalizations for a system of 3×3 matrices.

This procedure yields a polynomial expression for I in terms of simpler invariants, provided that I is reducible.

In each step of this reduction the equations employed are homogeneous throughout in the elements of the constituent matrices. In particular, in applying the process (ii), if the given invariant of the 3×3 system is of degrees $l, m, n, \dots$ in the matrices $\mathbf{a}, \mathbf{b}, \mathbf{c}, \dots$ respectively, then each term of the relation connecting it with the invariants of the 2×2 system is also of degrees $l, m, n, \dots$ in the elements of $\mathbf{a}, \mathbf{b}, \mathbf{c}, \dots$. A similar remark applies to the converse procedure (iv); the relations of § 7 illustrate this feature. Since each invariant of the 2×2 system can be expressed as a polynomial in the invariants (6.6), and these invariants are all of the fourth degree or less in the elements of the constituent matrices, it follows that any invariant of the 3×3 system which is of degree greater than the fourth is reducible. Furthermore, each irreducible invariant of the 3×3 system must be such that it arises as a result of applying the symbolic process to one or other of the invariants (6.6). Invariants involving two matrices have been examined in § 7. For the remaining invariants of (6.6) we have

$$\begin{aligned} a_{1\alpha} c_{\alpha\beta} b_{\beta 1} &= a_{1i} c_{ij} b_{j1} - a_{11} c_{1i} b_{i1} - b_{11} a_{1i} c_{i1} + a_{11} b_{11} c_{11}, \\ a_{1\alpha} c_{\alpha\beta} d_{\beta\gamma} b_{\gamma 1} &= a_{1i} c_{ij} d_{jk} b_{k1} - a_{11} c_{1i} d_{ij} b_{j1} - b_{11} a_{1i} c_{ij} d_{j1} - a_{1i} c_{i1} d_{1k} b_{k1} + \\ &\quad + a_{11} c_{11} d_{1i} b_{i1} + a_{11} b_{11} c_{1i} d_{i1} + b_{11} d_{11} a_{1i} c_{i1} - a_{11} b_{11} c_{11} d_{11}. \end{aligned} \quad (8.1)$$

From these relations it follows that the only invariants of the 3×3 system of matrices involving three or four different matrices which are not reducible are those of the types derived by taking the leading terms of (5.1) (I), (II), (III) and (5.17). If $c_{\alpha\beta} - d_{\alpha\beta}$ the left hand member of the second of equations (8.1) is reducible by means of the Cayley Hamilton theorem and this is sufficient to exclude the invariant formed by taking the leading term $[a_i a_k^2 a_j]_{11}$ of the third product of (III) in (5.1). Thus

$$\begin{aligned} [a b^2 c]_{11} &= a_{1i} b_{ij} b_{jk} c_{k1} \\ &= a_{1\alpha} b_{\alpha\beta} b_{\beta\gamma} c_{\gamma 1} + a_{11} b_{1\alpha} b_{\alpha\beta} c_{\beta 1} + c_{11} a_{1\alpha} b_{\alpha\beta} b_{\beta 1} + \\ &\quad + a_{1\alpha} b_{\alpha 1} b_{1\beta} c_{\beta 1} + a_{11} b_{11} b_{1\alpha} c_{\alpha 1} + a_{11} c_{11} b_{1\alpha} b_{\alpha 1} + \\ &\quad + b_{11} c_{11} a_{1\alpha} b_{\alpha 1} + a_{11} b_{11}^2 c_{11}, \end{aligned} \quad (8.2)$$

and

$$\begin{aligned} a_{1\alpha} b_{\alpha\beta} b_{\beta\gamma} c_{\gamma 1} &= a_{1\alpha} [b_{\alpha\gamma} b_{\beta\beta} + \tfrac{1}{2} \delta_{\alpha\gamma} \{b_{\beta\lambda} b_{\lambda\beta} - (b_{\beta\beta})^2\}] c_{\gamma 1} \\ &= b_{\beta\beta} a_{1\alpha} b_{\alpha\gamma} c_{\gamma 1} + \tfrac{1}{2} a_{1\alpha} c_{\alpha 1} [b_{\beta\lambda} b_{\lambda\beta} - (b_{\beta\beta})^2]. \end{aligned} \quad (8.3)$$

Again, invariants such as

$$\text{tr } \mathbf{a} \mathbf{b} \mathbf{c}, \quad \text{tr } \mathbf{a} \mathbf{b} \mathbf{c}^2, \quad \text{tr } \mathbf{a} \mathbf{b} \mathbf{c} \mathbf{d},$$

which form part of the integrity basis for an isotropic system are all expressible in terms of simpler isotropic and transversely isotropic invariants by using stages (ii), (iii), (iv) of the reduction process described above.

Summarizing the results of § 7 and the conclusions of the present section we find that any polynomial in the elements of the r symmetric 3×3 matrices $\mathbf{a}_1, \mathbf{a}_2, \dots, \mathbf{a}_r$ which is invariant under the group of transformations (4.1) is expressible as a polynomial in the systems of invariants

$$\begin{aligned} &\text{tr } \mathbf{a}_i, \quad \text{tr } \mathbf{a}_i^2, \quad \text{tr } \mathbf{a}_i^3, & (i) \\ &\text{tr } \mathbf{a}_i \mathbf{a}_j, \quad \text{tr } \mathbf{a}_i^2 \mathbf{a}_j, \quad \text{tr } \mathbf{a}_i \mathbf{a}_j^2, \quad \text{tr } \mathbf{a}_i^2 \mathbf{a}_j^2, & (ii) \\ &[\mathbf{a}_i]_{11}, \quad [\mathbf{a}_i^2]_{11}, & (iii) \\ &[\mathbf{a}_i \mathbf{a}_j]_{11}, \quad [\mathbf{a}_i^2 \mathbf{a}_j]_{11}, \quad [\mathbf{a}_i \mathbf{a}_j^2]_{11} \quad (i \neq j), & (iv) \\ &[\mathbf{a}_i \mathbf{a}_j \mathbf{a}_k]_{11}, \quad [\mathbf{a}_j \mathbf{a}_i \mathbf{a}_k]_{11}, \quad [\mathbf{a}_i \mathbf{a}_j \mathbf{a}_k^2]_{11}, & (v) \\ &[\mathbf{a}_i \mathbf{a}_k \mathbf{a}_j^2]_{11}, \quad [\mathbf{a}_j \mathbf{a}_k \mathbf{a}_i^2]_{11}, \quad (i, j, k \text{ all different}), & (8.4) \\ &[\mathbf{a}_i \mathbf{a}_j \mathbf{a}_k \mathbf{a}_l]_{11}, \quad [\mathbf{a}_i \mathbf{a}_j \mathbf{a}_l \mathbf{a}_k]_{11}, \quad [\mathbf{a}_i \mathbf{a}_k \mathbf{a}_j \mathbf{a}_l]_{11}, \\ &[\mathbf{a}_i \mathbf{a}_k \mathbf{a}_l \mathbf{a}_j]_{11}, \quad [\mathbf{a}_i \mathbf{a}_l \mathbf{a}_j \mathbf{a}_k]_{11}, \quad [\mathbf{a}_j \mathbf{a}_i \mathbf{a}_k \mathbf{a}_l]_{11}, & (vi) \\ &(i, j, k, l \text{ all different}), \end{aligned}$$

obtained by taking all possible choices of one, two, three or four of the letters i, j, k, l , as appropriate, from the numbers 1 to r . In this scheme redundant products of type (v) are excluded by using (5.8).

The invariants for transversely isotropic bodies are derived from (8.4) by replacing $\mathbf{a}_p$ ($p = 1, 2, \dots, n+1$) by the kinematic matrices $\mathbf{E}$, $\mathbf{A}_p$ ($p = 1, 2, \dots, n$). The coefficients in the stress deformation relations follow by forming from (8.4) invariants which involve the transformation matrix $\mathbf{C}$ linearly. In this way we obtain the system of coefficients

$$\begin{aligned} &\text{tr } \mathbf{C}, \quad \text{tr } \mathbf{C} \mathbf{a}_i, \quad \text{tr } \mathbf{C} \mathbf{a}_i^2, & (i) \\ &[\mathbf{C}]_{11}, \quad [\mathbf{C} \mathbf{a}_i]_{11}, \quad [\mathbf{C} \mathbf{a}_i^2]_{11}, & (ii) \\ &[\mathbf{C} \mathbf{a}_i \mathbf{a}_j]_{11}, \quad [\mathbf{C} \mathbf{a}_j \mathbf{a}_i]_{11}, \quad [\mathbf{C} \mathbf{a}_i \mathbf{a}_j^2]_{11}, \quad [\mathbf{C} \mathbf{a}_j \mathbf{a}_i^2]_{11}, & (iii) \\ &[\mathbf{C} \mathbf{a}_i \mathbf{a}_j \mathbf{a}_k]_{11}, \quad [\mathbf{C} \mathbf{a}_i \mathbf{a}_k \mathbf{a}_j]_{11}, \quad [\mathbf{C} \mathbf{a}_j \mathbf{a}_i \mathbf{a}_k]_{11}, & (8.5) \\ &[\mathbf{C} \mathbf{a}_j \mathbf{a}_k \mathbf{a}_i]_{11}, \quad [\mathbf{C} \mathbf{a}_k \mathbf{a}_i \mathbf{a}_j]_{11}, \quad [\mathbf{a}_i \mathbf{C} \mathbf{a}_j \mathbf{a}_k]_{11}, & (iv) \\ &(i \neq j \neq k \neq i) \end{aligned}$$

in which $\mathbf{a}_i, \mathbf{a}_j, \mathbf{a}_k$ can represent any of the kinematic matrices $\mathbf{E}$, $\mathbf{A}_p$ ($p = 1, 2, \dots, n$). The coefficients (i) in (8.5) are those appropriate to isotropic materials

and may be written in more familiar forms in terms of the convected system ϑ^i . For we have

$$\begin{aligned}\text{tr } \mathbf{C} &= \frac{\partial \vartheta^i}{\partial x^r} \frac{\partial \vartheta^j}{\partial x^r} = \gamma^{ij}, \\ \text{tr } \mathbf{C} \mathbf{a}_p &= \frac{1}{2} \left(\frac{\partial \vartheta^i}{\partial x^r} \frac{\partial \vartheta^j}{\partial x^s} + \frac{\partial \vartheta^i}{\partial x^s} \frac{\partial \vartheta^j}{\partial x^r} \right) {}^{(p)}a_{rs} = {}^{(p)}\alpha^{ij}, \\ \text{tr } \mathbf{C} \mathbf{a}_p^2 &= \frac{1}{2} \left(\frac{\partial \vartheta^i}{\partial x^r} \frac{\partial \vartheta^j}{\partial x^s} + \frac{\partial \vartheta^i}{\partial x^s} \frac{\partial \vartheta^j}{\partial x^r} \right) {}^{(p)}a_{st} {}^{(p)}a_{tr} = {}^{(p)}\alpha_t^i {}^{(p)}\alpha^{tj},\end{aligned}\quad (8.6)$$

where ${}^{(p)}\alpha_j^i$, ${}^{(p)}\alpha^{ij}$ may represent the mixed and contravariant components formed from any of the kinematic tensors η_{ij} , $\alpha_{ij}^{(r)}$ with the metric tensor γ^{ij} , and ${}^{(p)}a_{ij}$ represent the components of the corresponding tensor (that is, the elements of the matrix $\mathbf{a}_p$) referred to the cartesian reference frame x_i . From (8.4) to (8.6) it follows that the stress tensor τ^{ij} for a transversely isotropic body may be written

$$\begin{aligned}\tau^{ij} &= C_{11}^{ij} \Phi_0 + C_{1r}^{ij} \sum_{l=1}^{n+1} [{}^{(l)}a_{r1} \Phi_l^{(1)} + {}^{(l)}a_{rs} {}^{(l)}a_{s1} \Phi_l^{(2)}] + \\ &+ C_{1r}^{ij} \sum_{l=1}^{n+1} \sum_{\substack{m=1 \\ l \neq m}}^{n+1} [{}^{(l)}a_{rs} {}^{(m)}a_{s1} \Phi_{lm}^{(1)} + {}^{(l)}a_{rs} {}^{(m)}a_{st} {}^{(m)}a_{t1} \Phi_{lm}^{(2)}] + \\ &+ C_{1r}^{ij} \sum_{l=1}^{n-1} \sum_{m=2}^n \sum_{\substack{p=3 \\ l < m < p}}^{n+1} [{}^{(l)}a_{rs} {}^{(m)}a_{st} {}^{(p)}a_{t1} \Phi_{lmp} + {}^{(l)}a_{rs} {}^{(p)}a_{st} {}^{(m)}a_{t1} \Phi_{lpm} + \\ &+ {}^{(m)}a_{rs} {}^{(l)}a_{st} {}^{(p)}a_{t1} \Phi_{mlp} + {}^{(m)}a_{rs} {}^{(p)}a_{st} {}^{(l)}a_{t1} \Phi_{mpl} + \\ &+ {}^{(p)}a_{rs} {}^{(l)}a_{st} {}^{(m)}a_{t1} \Phi_{plm}] + \\ &+ C_{rs}^{ij} \sum_{l=1}^{n-1} \sum_{m=2}^n \sum_{\substack{p=3 \\ l < m < p}}^{n+1} ({}^{(l)}a_{1r} {}^{(m)}a_{st} {}^{(p)}a_{t1} \Phi'_{lmp} + \\ &+ \gamma^{ij} \Psi_0 + \sum_{l=1}^{n+1} ({}^{(l)}\alpha^{ij} \Psi_l^{(1)} + \sum_{l=1}^{n+1} ({}^{(l)}\alpha_h^i {}^{(l)}\alpha^{kj} \Psi_l^{(2)}),\end{aligned}\quad (8.7)$$

where Φ_0 , $\Phi_l^{(k)}$, $\Phi_m^{(k)}$, Φ_{uvv} , Φ'_{lmp} , Ψ_0 , $\Psi_l^{(k)}$ are polynomial invariant functions of the invariants derived from (8.4) by substituting the kinematic matrices $\mathbf{E}$, $\mathbf{A}_r$ for $\mathbf{a}_p$. By excluding from (8.7) the terms involving Φ_0 , $\Phi_l^{(k)}$, $\Phi_m^{(k)}$, Φ_{uvv} and Φ'_{lmp} and restricting Ψ_0 , $\Psi_l^{(k)}$ to be functions only of the invariants (i) and (ii) in (8.4) we derive forms appropriate to isotropic bodies. These are not, however, the most general forms for the isotropic case, since by the procedure of §§ 7 and 8 some of the isotropic coefficients and invariants have been expressed partly in terms of those peculiar to transversely isotropic materials.

9. Elastic materials

We consider an elastic material whose elastic properties are described in the usual way by means of a strain energy function W which is a polynomial in the strain components e_{ij} . For such materials we have (GREEN & ZERNA (1954))

$$\ell W = \frac{1}{2\sqrt{I}} \left(\frac{\partial W}{\partial e_{ij}} + \frac{\partial W}{\partial e_{ji}} \right), \quad (9.1)$$

where

$$I = |\delta_{ij} + 2e_{ij}|. \quad (9.2)$$

From the formulation of §§ 4 and 5 we deduce that W can be expressed as a polynomial in the quantities I_r of (7.2) with e_{ij} replacing a_{ij} . Using (2.7) and (2.8) we then obtain

$$\begin{aligned} \tau^{ij} \sqrt{I} = & \gamma^{ij} \frac{\partial W}{\partial I_1} + 2\eta^{ij} \frac{\partial W}{\partial I_2} + 3\eta_r^i \eta^{rj} \frac{\partial W}{\partial I_3} + \\ & + \frac{\partial \theta^i}{\partial x^1} \frac{\partial \theta^j}{\partial x^1} \frac{\partial W}{\partial I_4} + \left(\frac{\partial \theta^i}{\partial x^r} \frac{\partial \theta^j}{\partial x^1} + \frac{\partial \theta^i}{\partial x^1} \frac{\partial \theta^j}{\partial x^r} \right) e_{r1} \frac{\partial W}{\partial I_5}. \end{aligned} \quad (9.3)$$

In the alternative formulation of § 6, the invariants K_r of § 7 replace I_r giving in place of (9.3)

$$\begin{aligned} \tau^{ij} \sqrt{I} = & \frac{\partial \theta^i}{\partial x^\alpha} \frac{\partial \theta^j}{\partial x^\alpha} \frac{\partial W}{\partial K_1} + \left(\frac{\partial \theta^i}{\partial x^\alpha} \frac{\partial \theta^j}{\partial x^\beta} + \frac{\partial \theta^i}{\partial x^\beta} \frac{\partial \theta^j}{\partial x^\alpha} \right) e_{\alpha\beta} \frac{\partial W}{\partial K_2} + \frac{\partial \theta^i}{\partial x^1} \frac{\partial \theta^j}{\partial x^1} \frac{\partial W}{\partial K_3} + \\ & + \left(\frac{\partial \theta^i}{\partial x^1} \frac{\partial \theta^j}{\partial x^\alpha} + \frac{\partial \theta^i}{\partial x^\alpha} \frac{\partial \theta^j}{\partial x^1} \right) \left(e_{\alpha 1} \frac{\partial W}{\partial K_4} + e_{\alpha\beta} e_{\beta 1} \frac{\partial W}{\partial K_5} \right) + \\ & + \frac{1}{2} \left(\frac{\partial \theta^i}{\partial x^\alpha} \frac{\partial \theta^j}{\partial x^\beta} + \frac{\partial \theta^i}{\partial x^\beta} \frac{\partial \theta^j}{\partial x^\alpha} \right) e_{1\alpha} e_{1\beta} \frac{\partial W}{\partial K_5}. \end{aligned} \quad (9.4)$$

Equation (9.3) may evidently be regarded as a special case of (8.7), while (9.4) corresponds to the form which may be derived from (6.6).

10. Orthotropic bodies

In a previous paper (1958) the writer has derived for orthotropic bodies the relation

$$\tau^{ij} = A_{it}^{ij} \Theta_{tt} + \varepsilon_{rst} \varepsilon_{rst} A_{rs}^{ij} [P_{rs;t}^{(1)} + P_{rs;t}^{(2)} + Q_{rs;t}^{(1)} + Q_{rs;t}^{(2)} + R_{rs;t}], \quad (10.1)$$

in which

$$P_{rs;t}^{(1)} = e_{rs} \Theta_{it}^{(1)}, \quad P_{rs;t}^{(2)} = a_{rs}^{(1)} \Theta_{it}^{(2)}, \quad (i) \quad (10.2)$$

$$Q_{rs;t}^{(1)} = e_{rt} e_{ts} \Theta_{it}^{(3)}, \quad Q_{rs;t}^{(2)} = a_{rt}^{(1)} a_{ts}^{(1)} \Theta_{it}^{(4)}, \quad (ii)$$

$$R_{rs;t} = e_{rt} a_{ts}^{(1)} \Theta_{it}^{(5)} + e_{st} a_{tr}^{(1)} \Theta_{it}^{(6)}, \quad (iii)$$

ε_{rst} is equal to $+1$ or -1 according as r, s, t is an even or odd permutation of $1, 2, 3$ and is zero otherwise and in (10.1) summation is carried out over the repeated suffixes r, s, t . The functions $\Theta_{it}, \Theta_{it}^{(k)}$ are invariant functions of the quantities

$$\begin{aligned} & e_{ii}, \quad a_{ii}^{(1)}, \\ & e_{ij} e_{ji}, \quad a_{ij}^{(1)} a_{ji}^{(1)}, \quad e_{ij} a_{ji}^{(1)}, \\ & e_{12} e_{23} e_{31}, \quad e_{ij} e_{jk} a_{ki}^{(1)}, \quad e_{ij} a_{jk}^{(1)} a_{ki}^{(1)}, \quad a_{12}^{(1)} a_{23}^{(1)} a_{31}^{(1)} \\ & (i, j, k = 1, 2, 3; i \neq j \neq k \neq i; i, j, k \text{ not summed}), \end{aligned} \quad (10.3)$$

such that $\Theta_{it}^{(5)}$ and $\Theta_{it}^{(6)}$ change into each other by interchange of r and s ($r \neq s \neq t \neq r$). The relation (10.1) is form invariant under the group of transformations

$$(\bar{x}_1, \bar{x}_2, \bar{x}_3) = (\pm x_1, \pm x_2, \pm x_3), \quad (10.4)$$

describing the symmetry properties of orthotropic materials and (10.3) is the invariant system appropriate to the case where τ^{ij} depends upon the two kinematic matrices e_{ij} , $a_{ij}^{(1)}$.

The derivation of (10.1) to (10.3) depends upon the fact that in any product of the elements e_{ij} , $a_{ij}^{(1)}$ which is form invariant under the transformation group (10.4), each of the suffixes 1, 2, 3 must occur an even number of times. If such products are arranged in the form

$$b_{ij}c_{jkl}d_{kl}\dots f_{mn}h_{ni} \quad (i, j, k, l, \dots, m, n \text{ not summed}), \quad (10.5)$$

with equal suffixes adjacent, the limited range of values 1, 2, 3 for the indices $i, j, \dots, m, n$ implies that any product (10.5) can be resolved into a product of the invariants (10.3) in which at most three of the elements e_{ij} , $a_{ij}^{(1)}$ are multiplied together. An analogous argument limits the coefficients in (10.2).

A similar procedure may be followed when τ^{ij} depends upon any number of kinematic matrices e_{ij} , $a_{ij}^{(r)}$ ($r=1, 2, \dots, n$). In this case the invariant system corresponding to (10.3) may be derived from

$$b_{12}c_{23}d_{31} \quad (10.6)$$

by replacing each of b_{ij} , c_{ij} , d_{ij} in all possible ways by e_{ij} , $a_{ij}^{(r)}$ ($r=1, 2, \dots, n$), δ_{ij} . In place of (10.2) we obtain a set of functions

$$\begin{aligned} P_{rs;t}^{(k)} &= b_{rs}\Theta_{it}^{(k)}, & (i) \\ Q_{rs;t}^{(k)} &= b_{rt}b_{ts}\Phi_{it}^{(k)}, & (ii) \\ R_{rs;t}^{(k)} &= b_{rt}c_{ts}\Psi_{it}^{(k)} + b_{st}c_{tr}\chi_{it}^{(k)} \quad (b_{rs} \neq c_{rs}), & (iii) \end{aligned} \quad (10.7)$$

($k=1, 2, 3, \dots$; $r \neq s \neq t \neq r$; t not summed),

in which b_{rs} , c_{rs} are replaced in all possible ways by the kinematic matrices e_{rs} , $a_{rs}^{(p)}$ ($p=1, 2, \dots, n$) and $\Theta_{it}^{(k)}$, $\Phi_{it}^{(k)}$, $\Psi_{it}^{(k)}$, $\chi_{it}^{(k)}$ are invariant functions of the invariants derived from (10.6) which are such that $\Psi_{it}^{(k)}$, $\chi_{it}^{(k)}$ are interchanged by interchange of r and s .

It should be noted that when any orthotropic invariant is expressed in terms of the basic system derived from (10.6), this resolution is not necessarily unique. For example, the invariant product

$$I = b_{12}c_{23}d_{31}f_{12}h_{23}k_{31},$$

can be written

$$\begin{aligned} I &= (b_{12}f_{12})(c_{23}h_{23}) \cdot (d_{31}k_{31}) \\ &= (b_{12}c_{23}d_{31}) \cdot (f_{12}h_{23}k_{31}) = (f_{12}c_{23}d_{31}) \cdot (b_{12}h_{23}k_{31}) \\ &= \dots, \end{aligned} \quad (10.8)$$

and even if all of the constituent elements in I are equal, the first form of I in (10.8) represents a different resolution from the remainder.

References

- ADKINS, J. E.: Phil. Trans. Roy. Soc. A **250**, 519 (1958).
GREEN, A. E., & R. S. RIVLIN: Arch. Rational Mech. Anal. **1**, 1 (1957).
GREEN, A. E., & W. ZERNA: Theoretical Elasticity. Oxford University Press 1954.
NOLL, W.: J. Rational Mech. Anal. **4**, 3 (1955).
OLDROYD, J. G.: Proc. Roy. Soc. A **200**, 523 (1950).
RIVLIN, R. S.: J. Rational Mech. Anal. **4**, 681 (1955).
RIVLIN, R. S., & J. L. ERICKSEN: J. Rational Mech. Anal. **4**, 323 (1955).
SPENCER, A. J. M., & R. S. RIVLIN: Arch. Rational Mech. Anal. **2**, 309 (1959a).
SPENCER, A. J. M., & R. S. RIVLIN: Arch. Rational Mech. Anal. **2**, 435 (1959b).
TRUESDELL, C.: Comm. Pure Appl. Math. **8**, 123 (1955a).
TRUESDELL, C.: J. Rational Mech. Anal. **4**, 83 (1955b).
TRUESDELL, C.: J. Rational Mech. Anal. **4**, 1019 (1955c).
TURNBULL, H. W.: The Theory of Determinants, Matrices and Invariants. London: Blackie and Son Ltd. 1928.
WEITZENBÖCK, R.: Invarianten-Theorie. Groningen 1923.

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Further Results in the Theory of Matrix Polynomials

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1. Introduction

The theory of matrix polynomials and its application to the mechanics of isotropic continua has been discussed in earlier papers [1], [2], [3], [4]. The purpose of this paper is to give some further results which extend and in some respects simplify the theory which was put forward in [3] and [4].

The following theorem was proven in [3]:

Theorem 1. *Any matrix polynomial $\mathbf{P}$ in R 3×3 matrices $\mathbf{a}_P$ ($P = 1, 2, \dots, R$) can be expressed as a matrix polynomial of lower or equal extension and total degree, in which*

- (i) *each matrix product is either the unit matrix $\mathbf{I}$ or is formed from some or all of the factors $\mathbf{a}_P$ ($P = 1, 2, \dots, R$) and at most two of the factors $\mathbf{a}_P^2$ ($P = 1, 2, \dots, R$);*
- (ii) *no two factors in a single matrix product are the same;*
- (iii) *each matrix product is of lower or equal partial degree in each of the matrices $\mathbf{a}_P$ ($P = 1, 2, \dots, R$) than the matrix polynomial $\mathbf{P}$;*
- (iv) *matrix products containing two of the factors $\mathbf{a}_P^2$ ($P = 1, 2, \dots, R$) contain them consecutively;*
- (v) *no matrix product containing both of the factors $\mathbf{a}_K^2$ and $\mathbf{a}_L^2$ contains either of the factors $\mathbf{a}_K$ or $\mathbf{a}_L$, unless it has the form $\mathbf{a}_K \mathbf{a}_L^2 \mathbf{a}_K$;*
- (vi) *$\mathbf{a}_K$ precedes $\mathbf{a}_K^2$ in any matrix product containing both $\mathbf{a}_K$ and $\mathbf{a}_K^2$ as factors.*

For definitions of the terms employed above and elsewhere in this paper, reference is made to [2] and [3].

The sets of matrix products which satisfy the conditions of Theorem 1 were enumerated in [3] for the cases $R = 1, 2, 3, 4$ and 5 . Also in [3], a finite integrity basis for R symmetric 3×3 matrices was determined, by a method based on Theorem 1.

To this point all results were derived by means of algebraic operations based ultimately on the Hamilton-Cayley theorem. At this stage Peano's theorem was introduced to further limit the number of elements in the integrity basis for R symmetric 3×3 matrices, and also the number of necessary terms in the closed expression for a symmetric isotropic matrix polynomial in R symmetric 3×3 matrices, and the terms of this expression were enumerated.

In this note a stronger version of Theorem 1 will be proven, and this will be used to construct a closed form expression for a matrix polynomial $\mathbf{P}$ (not necessarily symmetric) in R 3×3 matrices. The matrix products in this expression will all be of degree ≤ 5 . For the case in which $\mathbf{P}$ is a symmetric matrix polynomial in R symmetric 3×3 matrices, it will be shown that $\mathbf{P}$ can be expressed in a closed form which contains fewer terms than the corresponding closed expression which was given in [3]; moreover, all the results of this paper will be obtained only by algebraic operations based on the Hamilton-Cayley theorem, and without use of Peano's theorem.

Then, it will be shown that the integrity basis for R symmetric 3×3 matrices, which was derived in [3] by the use of Peano's theorem, and further discussed in [4] for the cases $R \leq 5$, can be obtained directly from the results of this paper. Thus, it will be shown that all the results of [3] and [4], and some further results, are direct consequences of theorems of matrix algebra, and do not depend for their proof on general results of the theory of algebraic invariants.

The application of these results to the formulation of constitutive equations describing the mechanical properties of isotropic materials for which the stress depends on a number of kinematic tensors is obvious. In § 11, they are applied to the case when the stress components are functionals of the displacement gradients, over the time interval from the instant at which the deformation commences to the instant of measurement of the stress. This problem has been discussed previously in [6].

2. Some results concerning matrix polynomials in 3×3 matrices

In this section some results, proofs of which were given in [2] and [3], will be summarised for convenient reference. The notation $\mathbf{Q}(r)$ will be introduced to denote an arbitrary matrix polynomial of total degree less than or equal to r in the matrices $\mathbf{a}_P$ ($P=1, 2, \dots, R$). If $\mathbf{P}$ is a matrix polynomial in these matrices, the statement $\mathbf{P} = \mathbf{Q}(r)$ is understood to mean " $\mathbf{P}$ can be expressed as a matrix polynomial of total degree less than or equal to r in the matrices $\mathbf{a}_P$ ($P=1, 2, \dots, R$)". Clearly

$$\mathbf{Q}(r) \mathbf{Q}(s) = \mathbf{Q}(r+s).$$

A similar notation was introduced by ADKINS [5].

In this notation, certain of the results given in [2] and [3] may be stated as follows. Let $\mathbf{x}$, $\mathbf{y}$ and $\mathbf{z}$ be matrix products (not equal to the unit matrix $\mathbf{I}$) of total degrees r , s and t respectively in the 3×3 matrices $\mathbf{a}_P$ ($P=1, 2, \dots, R$). Then

$$\mathbf{x} \mathbf{y} \mathbf{z} + \mathbf{x} \mathbf{z} \mathbf{y} + \mathbf{y} \mathbf{x} \mathbf{z} + \mathbf{y} \mathbf{z} \mathbf{x} + \mathbf{z} \mathbf{x} \mathbf{y} + \mathbf{z} \mathbf{y} \mathbf{x} = \mathbf{Q}(r+s+t-1), \quad (2.1)$$

$$\mathbf{x}^2 \mathbf{y} + \mathbf{y} \mathbf{x}^2 + \mathbf{x} \mathbf{y} \mathbf{x} = \mathbf{Q}(2r+s-1), \quad (2.2)$$

$$\mathbf{x}^2 \mathbf{y} \mathbf{x} + \mathbf{x} \mathbf{y} \mathbf{x}^2 = \mathbf{Q}(3r+s-1), \quad (2.3)$$

$$\mathbf{x}^2 \mathbf{y} \mathbf{z}^2 = \mathbf{Q}(2r+s+2t-1), \quad (2.4)$$

$$\mathbf{x} \mathbf{y}^2 \mathbf{x}^2 \mathbf{z} = \mathbf{Q}(3r+2s+t-1). \quad (2.5)$$

In §§ 3, 4 and 5, we apply these formulas to show that matrix products of the forms

$$(i) \quad a_K a_L a_M^2 a_N^2, \quad a_K a_M^2 a_N^2 a_L, \quad a_M^2 a_N^2 a_K a_L; \quad (2.6)$$

$$(ii) \quad a_K a_L a_M a_N a_S^2, \quad a_K a_L a_M a_S^2 a_N, \quad a_K a_L a_S^2 a_M a_N, \\ a_K a_S^2 a_L a_M a_N, \quad a_S^2 a_K a_L a_M a_N; \quad (2.7)$$

$$(iii) \quad a_K a_L a_M a_N a_S a_T \quad (2.8)$$

may be expressed as matrix polynomials of degree five or less in the matrices $a_K, a_L, a_M, a_N, a_S, a_T$.

3. Matrix products $a_K a_L a_M^2 a_N^2$, $a_K a_M^2 a_N^2 a_L$, $a_M^2 a_N^2 a_K a_L$

In (2.1), we replace x, y, z by a_K, a_L, a_N^2 respectively and multiply the relation so obtained on the left by a_M^2 . This gives

$$a_M^2 (a_K a_L a_N^2 + a_K a_N^2 a_L + a_L a_K a_N^2 + a_L a_N^2 a_K + a_N^2 a_K a_L + a_N^2 a_L a_K) = Q(5). \quad (3.1)$$

Now, by relations of the type (2.4), multiplied on the right by appropriate factors where necessary, each of the products $a_M^2 a_K a_N^2 a_L$, $a_M^2 a_L a_K a_N^2$, $a_M^2 a_K a_L a_N^2$ and $a_M^2 a_L a_N^2 a_K$ can be expressed as a matrix polynomial of degree five or less in the matrices a_K, a_L, a_M and a_N . Inserting these results into (3.1), it follows that

$$a_M^2 a_N^2 a_K a_L + a_M^2 a_N^2 a_L a_K = Q(5). \quad (3.2)$$

In a somewhat similar manner, replacing x, y and z in (2.1) by a_K, a_M^2 , and a_N^2 respectively, multiplying the relation so obtained on the right by a_L , and using relations of the type (2.4) where possible, it is found that

$$a_M^2 a_N^2 a_K a_L + a_N^2 a_M^2 a_K a_L + a_K a_M^2 a_N^2 a_L + a_K a_N^2 a_M^2 a_L = Q(5). \quad (3.3)$$

Next, in (2.1), replace x by $a_L a_M^2$, y by a_N^2 , and z by a_K , obtaining

$$a_L a_M^2 a_N^2 a_K + a_L a_M^2 a_K a_N^2 + a_N^2 a_L a_M^2 a_K + a_N^2 a_K a_L a_M^2 + \\ + a_K a_L a_M^2 a_N^2 + a_K a_N^2 a_L a_M^2 = Q(5). \quad (3.4)$$

By relations of the type (2.4), multiplied on the left or right by appropriate factors where necessary, each of the matrix products $a_L a_M^2 a_K a_N^2$, $a_N^2 a_L a_M^2 a_K$, $a_N^2 a_K a_L a_M^2$ and $a_K a_N^2 a_L a_M^2$ can be expressed as a matrix polynomial of degree less than or equal to five in the matrices a_K, a_L, a_M and a_N . Inserting these results in (3.4) gives the relation

$$a_L a_M^2 a_N^2 a_K + a_K a_L a_M^2 a_N^2 = Q(5). \quad (3.5)$$

In a similar way, replacing x, y and z in (2.1) by $a_N^2 a_L$, a_M^2 and a_K respectively, it may be shown that

$$a_K a_M^2 a_N^2 a_L + a_M^2 a_N^2 a_L a_K = Q(5). \quad (3.6)$$

From (3.3), (3.6), and the relation obtained from (3.6) by interchanging a_M and a_N

$$a_M^2 a_N^2 a_K a_L - a_M^2 a_N^2 a_L a_K + a_N^2 a_M^2 a_K a_L - a_N^2 a_M^2 a_L a_K = Q(5). \quad (3.7)$$

Hence, from (3.2), the relation obtained from (3.2) by interchanging $\mathbf{a}_M$ and $\mathbf{a}_N$, and (3.7), we obtain

$$\mathbf{a}_M^2 \mathbf{a}_N^2 \mathbf{a}_K \mathbf{a}_L + \mathbf{a}_N^2 \mathbf{a}_M^2 \mathbf{a}_K \mathbf{a}_L = \mathcal{Q}(5). \quad (3.8)$$

Similarly, it may be shown that

$$\mathbf{a}_K \mathbf{a}_L \mathbf{a}_M^2 \mathbf{a}_N^2 + \mathbf{a}_K \mathbf{a}_L \mathbf{a}_N^2 \mathbf{a}_M^2 = \mathcal{Q}(5). \quad (3.9)$$

Next, in (2.5), replace $\mathbf{x}, \mathbf{y}$ and $\mathbf{z}$ by $\mathbf{a}_K + \mathbf{a}_N$, $\mathbf{a}_M$ and $\mathbf{a}_L$ respectively. Then

$$(\mathbf{a}_K + \mathbf{a}_N) \mathbf{a}_M^2 (\mathbf{a}_K + \mathbf{a}_N)^2 \mathbf{a}_L = \mathcal{Q}(5).$$

Expanding, and using the relations (of type (2.5)) $\mathbf{a}_K \mathbf{a}_M^2 \mathbf{a}_K^2 \mathbf{a}_L = \mathcal{Q}(5)$, and $\mathbf{a}_N \mathbf{a}_M^2 \mathbf{a}_N^2 \mathbf{a}_L = \mathcal{Q}(5)$, this last relation becomes

$$\begin{aligned} \mathbf{a}_K \mathbf{a}_M^2 \mathbf{a}_K \mathbf{a}_N \mathbf{a}_L + \mathbf{a}_K \mathbf{a}_M^2 \mathbf{a}_N \mathbf{a}_K \mathbf{a}_L + \mathbf{a}_N \mathbf{a}_M^2 \mathbf{a}_K \mathbf{a}_N \mathbf{a}_L + \mathbf{a}_N \mathbf{a}_M^2 \mathbf{a}_N \mathbf{a}_K \mathbf{a}_L + \\ + \mathbf{a}_K \mathbf{a}_M^2 \mathbf{a}_N^2 \mathbf{a}_L + \mathbf{a}_N \mathbf{a}_M^2 \mathbf{a}_K^2 \mathbf{a}_L = \mathcal{Q}(5). \end{aligned} \quad (3.10)$$

Now, replace $\mathbf{a}_N$ in (3.10) by $-\mathbf{a}_N$, and add the relation so obtained to (3.10). This gives

$$\mathbf{a}_K \mathbf{a}_M^2 \mathbf{a}_N^2 \mathbf{a}_L + \mathbf{a}_N \mathbf{a}_M^2 \mathbf{a}_K \mathbf{a}_N \mathbf{a}_L + \mathbf{a}_N \mathbf{a}_M^2 \mathbf{a}_N \mathbf{a}_K \mathbf{a}_L = \mathcal{Q}(5). \quad (3.11)$$

By relations readily derived from (2.2) and (2.4), and by (3.8)

$$\mathbf{a}_N \mathbf{a}_M^2 \mathbf{a}_K \mathbf{a}_N \mathbf{a}_L = -\mathbf{a}_N^2 \mathbf{a}_M^2 \mathbf{a}_K \mathbf{a}_L - \mathbf{a}_M^2 \mathbf{a}_K \mathbf{a}_N^2 \mathbf{a}_L + \mathcal{Q}(5) = -\mathbf{a}_N^2 \mathbf{a}_M^2 \mathbf{a}_K \mathbf{a}_L + \mathcal{Q}(5), \quad (3.12)$$

and

$$\mathbf{a}_N \mathbf{a}_M^2 \mathbf{a}_N \mathbf{a}_K \mathbf{a}_L = -\mathbf{a}_N^2 \mathbf{a}_M^2 \mathbf{a}_K \mathbf{a}_L - \mathbf{a}_M^2 \mathbf{a}_N^2 \mathbf{a}_K \mathbf{a}_L + \mathcal{Q}(5) = \mathcal{Q}(5). \quad (3.13)$$

Inserting (3.12) and (3.13) in (3.11), the relation

$$\mathbf{a}_K \mathbf{a}_M^2 \mathbf{a}_N^2 \mathbf{a}_L - \mathbf{a}_N^2 \mathbf{a}_M^2 \mathbf{a}_K \mathbf{a}_L = \mathcal{Q}(5) \quad (3.14)$$

is obtained. Hence, from (3.6) and (3.14)

$$\mathbf{a}_M^2 \mathbf{a}_N^2 \mathbf{a}_L \mathbf{a}_K + \mathbf{a}_N^2 \mathbf{a}_M^2 \mathbf{a}_K \mathbf{a}_L = \mathcal{Q}(5). \quad (3.15)$$

However, from (3.2) and (3.8),

$$\mathbf{a}_M^2 \mathbf{a}_N^2 \mathbf{a}_L \mathbf{a}_K - \mathbf{a}_N^2 \mathbf{a}_M^2 \mathbf{a}_K \mathbf{a}_L = \mathcal{Q}(5), \quad (3.16)$$

and it therefore follows that

$$\mathbf{a}_M^2 \mathbf{a}_N^2 \mathbf{a}_L \mathbf{a}_K = \mathcal{Q}(5). \quad (3.17)$$

From (3.17), (3.5), (3.6), and relations obtained from these by permuting the matrices, it is readily shown that

$$\mathbf{a}_M^2 \mathbf{a}_N^2 \mathbf{a}_K \mathbf{a}_L = \mathcal{Q}(5), \quad \mathbf{a}_K \mathbf{a}_M^2 \mathbf{a}_N^2 \mathbf{a}_L = \mathcal{Q}(5), \quad \mathbf{a}_K \mathbf{a}_L \mathbf{a}_M^2 \mathbf{a}_N^2 = \mathcal{Q}(5). \quad (3.18)$$

We thus have

Lemma 1. Any matrix product of the form $\mathbf{a}_M^2 \mathbf{a}_N^2 \mathbf{a}_K \mathbf{a}_L$, $\mathbf{a}_K \mathbf{a}_M^2 \mathbf{a}_N^2 \mathbf{a}_L$ or $\mathbf{a}_K \mathbf{a}_L \mathbf{a}_M^2 \mathbf{a}_N^2$ is expressible as a matrix polynomial of total degree five in the matrices $\mathbf{a}_K, \mathbf{a}_L, \mathbf{a}_M, \mathbf{a}_N$.

4. Matrix products $a_K a_L a_M a_N a_S^2$, $a_K a_L a_M a_S^2 a_N$, $a_K a_L a_S^2 a_M a_N$,
 $a_K a_S^2 a_L a_M a_N$, $a_S^2 a_K a_L a_M a_N$

Interchanging a_M and a_N in the first of the relations (3.18) and adding the relation so obtained to (3.18), we obtain

$$(a_M^2 a_N^2 + a_N^2 a_M^2) a_K a_L = Q(5).$$

Using a relation of the type (2.2), this leads to

$$a_M a_N^2 a_M a_K a_L = Q(5).$$

Applying similar considerations to the second and third of the relations (3.18), the three relations (3.18) yield

$$a_M a_N^2 a_M a_K a_L = Q(5), \quad a_K a_M a_N^2 a_M a_L = Q(5), \quad a_K a_L a_M a_N^2 a_M = Q(5). \quad (4.1)$$

Also, from relations of the type (2.4),

$$a_M^2 a_K a_N^2 a_L = Q(5), \quad a_M^2 a_K a_L a_N^2 = Q(5), \quad a_K a_M^2 a_L a_N^2 = Q(5). \quad (4.2)$$

In the first of the relations (3.18) replace a_M by a_S and a_N by $a_M + a_N$. Then

$$a_S^2 (a_M^2 + a_M a_N + a_N a_M + a_N^2) a_K a_L = Q(5). \quad (4.3)$$

By relations of the type (3.18), $a_S^2 a_M^2 a_K a_L = Q(5)$ and $a_S^2 a_N^2 a_K a_L = Q(5)$. Hence (4.3) yields

$$a_S^2 (a_M a_N + a_N a_M) a_K a_L = Q(5). \quad (4.4)$$

Alternatively, replacing a_M by $a_M + a_N$, and a_N by a_S in the first of (3.18), and proceeding in a manner similar to that used in deriving (4.4), the relation

$$(a_M a_N + a_N a_M) a_S^2 a_K a_L = Q(5), \quad (4.5)$$

may be obtained. In a similar manner, we obtain from the second and third of the relations (3.18) the four additional relations

$$\begin{aligned} a_K a_S^2 (a_M a_N + a_N a_M) a_L &= Q(5), & a_K (a_M a_N + a_N a_M) a_S^2 a_L &= Q(5), \\ a_K a_L a_S^2 (a_M a_N + a_N a_M) &= Q(5), & a_K a_L (a_M a_N + a_N a_M) a_S^2 &= Q(5). \end{aligned} \quad (4.6)$$

Again, replacing a_M by $a_M + a_N$ and a_N by a_S in (4.1) and employing relations of the types (2.2) and (3.18), we obtain

$$\begin{aligned} (a_M a_S^2 a_N + a_N a_S^2 a_M) a_K a_L &= Q(5), \\ a_K (a_M a_S^2 a_N + a_N a_S^2 a_M) a_L &= Q(5) \end{aligned} \quad (4.7)$$

and

$$a_K a_L (a_M a_S^2 a_N + a_N a_S^2 a_M) = Q(5).$$

Replacing a_M by a_S and a_N by $a_M + a_N$ in (4.2) and employing relations of the types (4.2), we obtain

$$\begin{aligned} a_S^2 a_K (a_M a_N + a_N a_M) a_L &= Q(5), \\ a_S^2 a_K a_L (a_M a_N + a_N a_M) &= Q(5) \end{aligned} \quad (4.8)$$

and

$$a_K a_S^2 a_L (a_M a_N + a_N a_M) = Q(5).$$

Similarly replacing $\mathbf{a}_N$ by $\mathbf{a}_S$ and $\mathbf{a}_M$ by $\mathbf{a}_M + \mathbf{a}_N$ in (4.2), we obtain, with (4.2),

$$\begin{aligned} (\mathbf{a}_M \mathbf{a}_N + \mathbf{a}_N \mathbf{a}_M) \mathbf{a}_K \mathbf{a}_S^2 \mathbf{a}_L &= \mathcal{Q}(5), \\ (\mathbf{a}_M \mathbf{a}_N + \mathbf{a}_N \mathbf{a}_M) \mathbf{a}_K \mathbf{a}_L \mathbf{a}_S^2 &= \mathcal{Q}(5) \\ \text{and} \quad \mathbf{a}_K (\mathbf{a}_M \mathbf{a}_N + \mathbf{a}_N \mathbf{a}_M) \mathbf{a}_L \mathbf{a}_S^2 &= \mathcal{Q}(5). \end{aligned} \quad (4.9)$$

The fifteen relations given in equations (4.4) to (4.9) lead immediately to the result that each of the five matrix products $\mathbf{a}_K \mathbf{a}_L \mathbf{a}_M \mathbf{a}_N \mathbf{a}_S^2, \dots, \mathbf{a}_S^2 \mathbf{a}_K \mathbf{a}_L \mathbf{a}_M \mathbf{a}_N$ is equal to the sum of the negative of the product formed from it by interchanging a pair of adjacent matrices in the sequence $\mathbf{a}_K, \mathbf{a}_L, \mathbf{a}_M, \mathbf{a}_N$ and a matrix polynomial of total degree five or less. From this result we obtain immediately

Lemma 2. *Any matrix product of 3×3 matrices $\mathbf{a}_P$ ($P = 1, 2, \dots, R$) which has the form $\mathbf{x} \mathbf{a}_S^2 \mathbf{y}$, where $\mathbf{x} \mathbf{y}$ is a matrix product of the form $\mathbf{a}_K \mathbf{a}_L \mathbf{a}_M \mathbf{a}_N$, is equal to any matrix product formed from $\mathbf{x} \mathbf{a}_S^2 \mathbf{y}$ by an even permutation of matrices in the sequence $\mathbf{a}_K, \mathbf{a}_L, \mathbf{a}_M, \mathbf{a}_N$ and to the negative of any matrix product formed from $\mathbf{x} \mathbf{a}_S^2 \mathbf{y}$ by an odd permutation of matrices in the sequence $\mathbf{a}_K, \mathbf{a}_L, \mathbf{a}_M, \mathbf{a}_N$, together, in each case, with a matrix polynomial of total degree five or less in the matrices $\mathbf{a}_P$.*

It will now be shown that each of the matrix products $\mathbf{a}_K \mathbf{a}_L \mathbf{a}_M \mathbf{a}_N \mathbf{a}_S^2, \dots, \mathbf{a}_S^2 \mathbf{a}_K \mathbf{a}_L \mathbf{a}_M \mathbf{a}_N$ can be expressed as a matrix polynomial of total degree five or less in the matrices $\mathbf{a}_P$. In (2.4), replace $\mathbf{x}, \mathbf{y}, \mathbf{z}$ by $\mathbf{a}_S, \mathbf{a}_K, \mathbf{a}_M + \mathbf{a}_N$ respectively. We obtain

$$\mathbf{a}_S^2 \mathbf{a}_K (\mathbf{a}_M^2 + \mathbf{a}_M \mathbf{a}_N + \mathbf{a}_N \mathbf{a}_M + \mathbf{a}_N^2) = \mathcal{Q}(4).$$

Since, by relations of the type (2.4), $\mathbf{a}_S^2 \mathbf{a}_K \mathbf{a}_M^2 = \mathcal{Q}(4)$ and $\mathbf{a}_S^2 \mathbf{a}_K \mathbf{a}_N^2 = \mathcal{Q}(4)$, this may be written

$$\mathbf{a}_S^2 \mathbf{a}_K (\mathbf{a}_M \mathbf{a}_N + \mathbf{a}_N \mathbf{a}_M) = \mathcal{Q}(4). \quad (4.10)$$

In (4.10), replace $\mathbf{a}_K$ by $\mathbf{a}_K \mathbf{a}_L$

$$\mathbf{a}_S^2 \mathbf{a}_K \mathbf{a}_L (\mathbf{a}_M \mathbf{a}_N + \mathbf{a}_N \mathbf{a}_M) = \mathcal{Q}(5). \quad (4.11)$$

Also in (4.10), replace $\mathbf{a}_M$ by $\mathbf{a}_L \mathbf{a}_M$

$$\mathbf{a}_S^2 \mathbf{a}_K (\mathbf{a}_L \mathbf{a}_M \mathbf{a}_N + \mathbf{a}_N \mathbf{a}_L \mathbf{a}_M) = \mathcal{Q}(5). \quad (4.12)$$

Again in (4.10), replace $\mathbf{a}_M$ by $\mathbf{a}_L$, and multiply the relation so obtained on the right by $\mathbf{a}_M$

$$\mathbf{a}_S^2 \mathbf{a}_K (\mathbf{a}_L \mathbf{a}_N + \mathbf{a}_N \mathbf{a}_L) \mathbf{a}_M = \mathcal{Q}(5). \quad (4.13)$$

Adding (4.11) to (4.12), and subtracting (4.13), the relation

$$\mathbf{a}_S^2 \mathbf{a}_K \mathbf{a}_L \mathbf{a}_M \mathbf{a}_N = \mathcal{Q}(5) \quad (4.14)$$

is obtained. In a similar way, it may be shown that

$$\mathbf{a}_K \mathbf{a}_L \mathbf{a}_M \mathbf{a}_N \mathbf{a}_S^2 = \mathcal{Q}(5). \quad (4.15)$$

Now, in (2.4), replace $\mathbf{x}$ by $\mathbf{a}_S^2$, $\mathbf{y}$ by $\mathbf{a}_K \mathbf{a}_L$ and $\mathbf{z}$ by $\mathbf{a}_M \mathbf{a}_N$, obtaining

$$\begin{aligned} \mathbf{a}_S^2 \mathbf{a}_K \mathbf{a}_L \mathbf{a}_M \mathbf{a}_N + \mathbf{a}_S^2 \mathbf{a}_M \mathbf{a}_N \mathbf{a}_K \mathbf{a}_L + \mathbf{a}_K \mathbf{a}_L \mathbf{a}_S^2 \mathbf{a}_M \mathbf{a}_N + \mathbf{a}_K \mathbf{a}_L \mathbf{a}_M \mathbf{a}_N \mathbf{a}_S^2 \\ + \mathbf{a}_M \mathbf{a}_N \mathbf{a}_S^2 \mathbf{a}_K \mathbf{a}_L + \mathbf{a}_M \mathbf{a}_N \mathbf{a}_K \mathbf{a}_L \mathbf{a}_S^2 = \mathcal{Q}(5). \end{aligned}$$

Inserting relations of the types (4.14) and (4.15), this becomes

$$\mathbf{a}_K \mathbf{a}_L \mathbf{a}_S^2 \mathbf{a}_M \mathbf{a}_N + \mathbf{a}_M \mathbf{a}_N \mathbf{a}_S^2 \mathbf{a}_K \mathbf{a}_L = \mathcal{Q}(5). \quad (4.16)$$

Since $\mathbf{a}_M \mathbf{a}_N \mathbf{a}_S^2 \mathbf{a}_K \mathbf{a}_L$ can be obtained from $\mathbf{a}_K \mathbf{a}_L \mathbf{a}_S^2 \mathbf{a}_M \mathbf{a}_N$ by an even permutation of the matrices $\mathbf{a}_K, \mathbf{a}_L, \mathbf{a}_M$ and $\mathbf{a}_N$ in the product $\mathbf{a}_K \mathbf{a}_L \mathbf{a}_M \mathbf{a}_N$, it follows from (4.16) and Lemma 2 that

$$\mathbf{a}_K \mathbf{a}_L \mathbf{a}_S^2 \mathbf{a}_M \mathbf{a}_N = \mathcal{Q}(5). \quad (4.17)$$

Next, in (2.1), replace $\mathbf{x}$ by $\mathbf{a}_S^2$, $\mathbf{y}$ by $\mathbf{a}_K \mathbf{a}_L \mathbf{a}_M$, and $\mathbf{z}$ by $\mathbf{a}_N$. Of the six products which appear in the resulting relation, four can be expressed as matrix polynomials of lower degree by means of relations of the types (4.14) and (4.15). The relation then becomes

$$\mathbf{a}_K \mathbf{a}_L \mathbf{a}_M \mathbf{a}_S^2 \mathbf{a}_N + \mathbf{a}_N \mathbf{a}_S^2 \mathbf{a}_K \mathbf{a}_L \mathbf{a}_M = \mathcal{Q}(5).$$

Applying Lemma 2, this relation may be written

$$\mathbf{a}_K \mathbf{a}_L \mathbf{a}_M \mathbf{a}_S^2 \mathbf{a}_N - \mathbf{a}_K \mathbf{a}_S^2 \mathbf{a}_L \mathbf{a}_M \mathbf{a}_N = \mathcal{Q}(5). \quad (4.18)$$

Again in (2.1), replace $\mathbf{x}$ by $\mathbf{a}_S^2$, $\mathbf{y}$ by $\mathbf{a}_K \mathbf{a}_L$, $\mathbf{z}$ by $\mathbf{a}_M$, and multiply the resulting relation on the right by $\mathbf{a}_N$. We obtain

$$\begin{aligned} (\mathbf{a}_S^2 \mathbf{a}_K \mathbf{a}_L \mathbf{a}_M + \mathbf{a}_S^2 \mathbf{a}_M \mathbf{a}_K \mathbf{a}_L + \mathbf{a}_K \mathbf{a}_L \mathbf{a}_S^2 \mathbf{a}_M + \mathbf{a}_K \mathbf{a}_L \mathbf{a}_M \mathbf{a}_S^2 \\ + \mathbf{a}_M \mathbf{a}_S^2 \mathbf{a}_K \mathbf{a}_L + \mathbf{a}_M \mathbf{a}_K \mathbf{a}_L \mathbf{a}_S^2) \mathbf{a}_N = \mathcal{Q}(5). \end{aligned}$$

Inserting relations of the types (4.14) and (4.17), we have

$$\mathbf{a}_K \mathbf{a}_L \mathbf{a}_M \mathbf{a}_S^2 \mathbf{a}_N + \mathbf{a}_M \mathbf{a}_S^2 \mathbf{a}_K \mathbf{a}_L \mathbf{a}_N + \mathbf{a}_M \mathbf{a}_K \mathbf{a}_L \mathbf{a}_S^2 \mathbf{a}_N = \mathcal{Q}(5).$$

With Lemma 2, this last relation may be written

$$2\mathbf{a}_K \mathbf{a}_L \mathbf{a}_M \mathbf{a}_S^2 \mathbf{a}_N + \mathbf{a}_K \mathbf{a}_S^2 \mathbf{a}_L \mathbf{a}_M \mathbf{a}_N = \mathcal{Q}(5). \quad (4.19)$$

Hence, from (4.18) and (4.19)

$$\mathbf{a}_K \mathbf{a}_S^2 \mathbf{a}_L \mathbf{a}_M \mathbf{a}_N = \mathcal{Q}(5), \quad \mathbf{a}_K \mathbf{a}_L \mathbf{a}_M \mathbf{a}_S^2 \mathbf{a}_N = \mathcal{Q}(5). \quad (4.20)$$

Equations (4.14), (4.15), (4.17) and (4.20) express

Lemma 3. Any matrix product of 3×3 matrices $\mathbf{a}_P$ ($P=1, 2, \dots, R$) which has the form $\mathbf{x} \mathbf{a}_S^2 \mathbf{y}$, where $\mathbf{x} \mathbf{y}$ is a matrix product of the form $\mathbf{a}_K \mathbf{a}_L \mathbf{a}_M \mathbf{a}_N$, is equal to a matrix polynomial of total degree five or less.

5. Matrix product $\mathbf{a}_K \mathbf{a}_L \mathbf{a}_M \mathbf{a}_N \mathbf{a}_S \mathbf{a}_T$

In (4.14), replace $\mathbf{a}_S$ by $\mathbf{a}_S + \mathbf{a}_T$. We obtain

$$(\mathbf{a}_S^2 + \mathbf{a}_S \mathbf{a}_T + \mathbf{a}_T \mathbf{a}_S + \mathbf{a}_T^2) \mathbf{a}_K \mathbf{a}_L \mathbf{a}_M \mathbf{a}_N = \mathcal{Q}(5).$$

Employing relations of the type (4.14), it follows that

$$(\mathbf{a}_S \mathbf{a}_T + \mathbf{a}_T \mathbf{a}_S) \mathbf{a}_K \mathbf{a}_L \mathbf{a}_M \mathbf{a}_N = \mathcal{Q}(5). \quad (5.1)$$

In a similar manner, each of the relations (4.15), (4.17) and (4.20) leads to a relation in which $\mathbf{a}_S^2$ in the original relation is replaced by $\mathbf{a}_S \mathbf{a}_T + \mathbf{a}_T \mathbf{a}_S$. Thus, $\mathbf{a}_K \mathbf{a}_L \mathbf{a}_M \mathbf{a}_N \mathbf{a}_S \mathbf{a}_T$ is equal to the sum of the negative of the matrix product formed from it by interchange of any two adjacent factors and a matrix polynomial of degree five or less. We obtain immediately

Lemma 4. *The matrix product $\mathbf{a}_K \mathbf{a}_L \mathbf{a}_M \mathbf{a}_N \mathbf{a}_S \mathbf{a}_T$ formed from six 3×3 matrices is equal to any matrix product formed from it by an even permutation of its factors and to the negative of any matrix product formed from it by an odd permutation of its factors, together, in each case, with a matrix polynomial of degree five or less in the matrices.*

To complete the discussion of this matrix product, in (2.1) replace $\mathbf{x}, \mathbf{y}$ and $\mathbf{z}$ by $\mathbf{a}_K \mathbf{a}_L$, $\mathbf{a}_M \mathbf{a}_N$ and $\mathbf{a}_S \mathbf{a}_T$ respectively. This gives

$$\begin{aligned} \mathbf{a}_K \mathbf{a}_L \mathbf{a}_M \mathbf{a}_N \mathbf{a}_S \mathbf{a}_T + \mathbf{a}_K \mathbf{a}_L \mathbf{a}_S \mathbf{a}_T \mathbf{a}_M \mathbf{a}_N + \mathbf{a}_M \mathbf{a}_N \mathbf{a}_K \mathbf{a}_L \mathbf{a}_S \mathbf{a}_T + \mathbf{a}_M \mathbf{a}_N \mathbf{a}_S \mathbf{a}_T \mathbf{a}_K \mathbf{a}_L + \\ + \mathbf{a}_S \mathbf{a}_T \mathbf{a}_K \mathbf{a}_L \mathbf{a}_M \mathbf{a}_N + \mathbf{a}_S \mathbf{a}_T \mathbf{a}_M \mathbf{a}_N \mathbf{a}_K \mathbf{a}_L = \mathcal{Q}(5). \end{aligned} \quad (5.2)$$

It is readily verified that each of the matrix products on the left hand side of (5.2) can be obtained from $\mathbf{a}_K \mathbf{a}_L \mathbf{a}_M \mathbf{a}_N \mathbf{a}_S \mathbf{a}_T$ by an even permutation of the factors. Hence, from Lemma 4 and (5.2)

$$\mathbf{a}_K \mathbf{a}_L \mathbf{a}_M \mathbf{a}_N \mathbf{a}_S \mathbf{a}_T = \mathcal{Q}(5). \quad (5.3)$$

This expresses

Lemma 5. *The matrix product $\mathbf{a}_K \mathbf{a}_L \mathbf{a}_M \mathbf{a}_N \mathbf{a}_S \mathbf{a}_T$ formed from six 3×3 matrices is equal to a matrix polynomial of degree five or less.*

We note that Lemma 3 and Lemma 1 may be obtained from Lemma 5 by taking appropriate pairs of the factors in the matrix product to be equal.

From Theorem 1 and Lemmas 1, 3 and 5, we obtain immediately

Lemma 6. *Any matrix product of degree six or more in the 3×3 matrices $\mathbf{a}_P$ ($P=1, 2, \dots, R$) may be expressed as a matrix polynomial of degree five or less in these matrices.*

An alternative derivation of Lemma 6 is given in § 12. Although this is shorter than the derivation given above, it fits into the general framework of the discussion somewhat less well.

6. Some matrix products of degree five

In this section we shall consider those matrix products which satisfy the conditions of Theorem 1, contain a factor $\mathbf{a}_K$ and a factor $\mathbf{a}_K^2$ and have total degree five or less. These have the forms

$$\mathbf{a}_K \mathbf{a}_L \mathbf{a}_K^2, \quad \mathbf{a}_K \mathbf{a}_L \mathbf{a}_K^2 \mathbf{a}_M, \quad \mathbf{a}_K \mathbf{a}_L \mathbf{a}_M \mathbf{a}_K^2, \quad \mathbf{a}_L \mathbf{a}_K \mathbf{a}_M \mathbf{a}_K^2. \quad (6.1)$$

In a manner similar to that in which (4.10) was derived, we obtain*

$$(\mathbf{a}_M \mathbf{a}_N + \mathbf{a}_N \mathbf{a}_M) \mathbf{a}_K \mathbf{a}_S^2 = \mathcal{Q}(4). \quad (6.2)$$

In (6.2), replace $\mathbf{a}_M, \mathbf{a}_N, \mathbf{a}_K$ and $\mathbf{a}_S$ by $\mathbf{a}_K, \mathbf{a}_L, \mathbf{a}_M$ and $\mathbf{a}_K$ respectively, obtaining the relation

$$\mathbf{a}_K \mathbf{a}_L \mathbf{a}_M \mathbf{a}_K^2 + \mathbf{a}_L \mathbf{a}_K \mathbf{a}_M \mathbf{a}_K^2 = \mathcal{Q}(4). \quad (6.3)$$

Also, by a relation of the type (2.3)

$$\mathbf{a}_K \mathbf{a}_L \mathbf{a}_K^2 \mathbf{a}_M = -\mathbf{a}_K^2 \mathbf{a}_L \mathbf{a}_K \mathbf{a}_M + \mathcal{Q}(4), \quad (6.4)$$

* Relations equivalent to (6.3) and (6.4) have been given by ADKINS [5]

and by a relation of the type (4.10)

$$a_K^2 a_L a_K a_M = -a_K^2 a_L a_M a_K + Q(4). \quad (6.5)$$

Further, by a relation similar to (2.3)

$$a_K^2 a_L a_M a_K = -a_K a_L a_M a_K^2 + Q(4), \quad (6.6)$$

and thus, from (6.4), (6.5) and (6.6)

$$a_K a_L a_K^2 a_M = -a_K a_L a_M a_K^2 + Q(4). \quad (6.7)$$

Equations (6.3) and (6.7) show that each of the matrix products (6.1) may be expressed as the sum of one or other of the products $a_K a_L a_K^2$ and $a_K a_L a_M a_K^2$ (with possibly the coefficient -1) and a matrix polynomial of lower total degree than that of the original matrix product.

Combining this result with Theorem 1 and Lemma 6, we obtain

Theorem 1'. *Any matrix polynomial in R , 3×3 matrices a_P ($P=1, 2, \dots, R$) may be expressed as a matrix polynomial of lower or equal extension and total degree, in which each matrix product satisfies the conditions (i) to (v) of Theorem 1 and in addition*

(vi') *each matrix product which contains both a_K and a_K^2 as factors has a_K as the first factor and a_K^2 as the last factor;*

(vii) *no matrix product has total degree greater than five in the matrices a_P ($P=1, 2, \dots, R$).*

7. Matrix polynomials in R 3×3 matrices

In this section Theorem 1' will be applied to obtain an explicit closed form expression for an arbitrary matrix polynomial in R 3×3 matrices. Enumerating all the matrix products which satisfy the conditions (i)–(v), (vi') and (vii) of Theorems 1 and 1', it is seen that any matrix polynomial P in R , 3×3 matrices a_P ($P=1, 2, \dots, R$) can be expressed as a matrix polynomial

$$P = \sum \chi_i \Pi_i,$$

where χ_i are scalar polynomial invariants of the matrices a_P , and Π_i are matrix products of the forms

$$\begin{aligned} &I; \quad a_K, \quad a_K^2; \quad a_K a_L, \quad a_K a_L^2, \quad a_K^2 a_L, \quad a_K^2 a_L^2, \quad a_K a_L a_K^2, \quad a_K a_L^2 a_K^2; \\ &a_K a_L a_M, \quad a_K^2 a_L a_M, \quad a_K a_L^2 a_M, \quad a_K a_L a_M^2, \quad a_K^2 a_L^2 a_M, \quad a_K a_L^2 a_M^2, \quad a_K a_L a_M a_K^2; \\ &a_K a_L a_M a_N, \quad a_K^2 a_L a_M a_N, \quad a_K a_L^2 a_M a_N, \quad a_K a_L a_M^2 a_N, \quad a_K a_L a_M a_N^2; \\ &a_K a_L a_M a_N a_S; \end{aligned} \quad (7.1)$$

and where K, L, M, N and S are integers, which are all different, and are chosen in all possible ways from the integers $1, 2, \dots, R$.

The sets of matrix products in (7.1) are included in the sets of matrix products which satisfy the conditions of Theorem 1, for $R \leq 5$, and which were enumerated in [3]. For $R=1$ and for $R=2$, the set of matrix products given by (7.1) is the same as the corresponding set given in [3]. For $R=3, 4$ or 5 , the set given by (7.1) contains fewer members than the corresponding set given in [3].

It may be noted that certain of the matrix products of the forms (7.4) can be expressed in terms of other matrix products of these forms. The arbitrary matrix polynomial $\mathbf{P}$ in R , 3×3 matrices $\mathbf{a}_P$ may therefore be expressed as a matrix polynomial involving fewer matrix products than are implied by (7.4). We shall, however, leave the discussion of such further limitations on the form of the matrix polynomial to a later paper.

8. Symmetric matrix polynomials in R symmetric 3×3 matrices

Suppose now that $\mathbf{P}$ is a symmetric matrix polynomial in symmetric 3×3 matrices $\mathbf{a}_P$ ($P = 1, 2, \dots, R$). Then

$$\mathbf{P} = \frac{1}{2}(\mathbf{P} + \mathbf{P}'),$$

where $\mathbf{P}'$ is the transpose of $\mathbf{P}$. Hence, by the results of § 7, $\mathbf{P}$ can be expressed in the form

$$\mathbf{P} = \frac{1}{2} \sum \chi_i (\mathbf{\Pi}_i + \mathbf{\Pi}'_i), \quad (8.1)$$

where χ_i and $\mathbf{\Pi}_i$ are defined as in § 7, and $\mathbf{\Pi}'_i$ is the transpose of $\mathbf{\Pi}_i$. Since the matrices $\mathbf{a}_P$ are symmetric matrices, $\mathbf{\Pi}'_i$ is derived from $\mathbf{\Pi}_i$ by reversing the order of the factors in $\mathbf{\Pi}_i$. Now, it follows from relations of the type (2.3) that

$$\begin{aligned} \text{and} \quad & \mathbf{a}_K \mathbf{a}_L \mathbf{a}_K^2 + \mathbf{a}_K^2 \mathbf{a}_L \mathbf{a}_K = \mathcal{Q}(3), \\ & \mathbf{a}_K \mathbf{a}_L^2 \mathbf{a}_K^2 + \mathbf{a}_K^2 \mathbf{a}_L^2 \mathbf{a}_K = \mathcal{Q}(4). \end{aligned} \quad (8.2)$$

From § 7, together with (8.1) and (8.2) it follows that $\mathbf{P}$ may be expressed as a sum, with scalar polynomial invariant coefficients, of terms of the forms

$$\begin{aligned} & \mathbf{I}; \quad \mathbf{a}_K, \quad \mathbf{a}_K^2; \quad \mathbf{a}_K \mathbf{a}_L + \mathbf{a}_L \mathbf{a}_K, \quad \mathbf{a}_K^2 \mathbf{a}_L + \mathbf{a}_L \mathbf{a}_K^2, \quad \mathbf{a}_K^2 \mathbf{a}_L^2 + \mathbf{a}_L^2 \mathbf{a}_K^2; \\ & \mathbf{a}_K \mathbf{a}_L \mathbf{a}_M + \mathbf{a}_M \mathbf{a}_L \mathbf{a}_K, \quad \mathbf{a}_K^2 \mathbf{a}_L \mathbf{a}_M + \mathbf{a}_M \mathbf{a}_L \mathbf{a}_K^2, \quad \mathbf{a}_K \mathbf{a}_L^2 \mathbf{a}_M + \mathbf{a}_M \mathbf{a}_L^2 \mathbf{a}_K, \\ & \mathbf{a}_K^2 \mathbf{a}_L^2 \mathbf{a}_M + \mathbf{a}_M \mathbf{a}_L^2 \mathbf{a}_K^2, \quad \mathbf{a}_K \mathbf{a}_L \mathbf{a}_M \mathbf{a}_K^2 + \mathbf{a}_K^2 \mathbf{a}_M \mathbf{a}_L \mathbf{a}_K; \quad (8.3) \\ & \mathbf{a}_K \mathbf{a}_L \mathbf{a}_M \mathbf{a}_N + \mathbf{a}_N \mathbf{a}_M \mathbf{a}_L \mathbf{a}_K, \quad \mathbf{a}_K^2 \mathbf{a}_L \mathbf{a}_M \mathbf{a}_N + \mathbf{a}_N \mathbf{a}_M \mathbf{a}_L \mathbf{a}_K^2, \quad \mathbf{a}_K \mathbf{a}_L^2 \mathbf{a}_M \mathbf{a}_N + \mathbf{a}_N \mathbf{a}_M \mathbf{a}_L^2 \mathbf{a}_K; \\ & \mathbf{a}_K \mathbf{a}_L \mathbf{a}_M \mathbf{a}_N \mathbf{a}_S + \mathbf{a}_S \mathbf{a}_N \mathbf{a}_M \mathbf{a}_L \mathbf{a}_K, \end{aligned}$$

for all selections of K, L, M, N and S (all different) chosen from the integers $1, 2, \dots, R$.

In [3] it was proven that a symmetric matrix polynomial $\mathbf{P}$ in symmetric 3×3 matrices $\mathbf{a}_P$ may be expressed as a sum, with scalar coefficients, of terms which were enumerated in Theorem 5 of [3]. The set of terms given by (8.3) is included in the set which was given in [3], but contains fewer members; the omission of certain terms which appear in Theorem 5 of [3] but not in (8.3) above is a consequence of (vi') and (vii) of Theorem 1' of this paper. Moreover, the set (8.3) has been derived from the Hamilton-Cayley theorem by methods of matrix algebra, and does not require the use of Peano's theorem, upon which Theorem 5 of [3] is partly based, for its derivation.

9. Invariants of R symmetric 3×3 matrices

It is easily verified that all of the results regarding the invariants under the orthogonal transformation group of R symmetric 3×3 matrices which were obtained in [3] and [4] can be obtained from Theorem 1' by a slight modification

of the arguments used in [3] and [4]. As shown in [3], any polynomial scalar invariant of the R symmetric 3×3 matrices $\mathbf{a}_P$ ($P=1, 2, \dots, R$) may be expressed as a polynomial in invariants of the form $\text{tr } \mathbf{a}_M \Pi_i$, where $\Pi_i + \Pi_i^*$ are the expressions (8.3). Π_i is of total degree less than or equal to five in the $\mathbf{a}_P$ and it follows that $\text{tr } \mathbf{a}_M \Pi_i$ is of total degree less than or equal to six in the $\mathbf{a}_P$. Thus every irreducible invariant of the symmetric 3×3 matrices $\mathbf{a}_P$ is of degree less than or equal to six in the $\mathbf{a}_P$.

Also, any invariant of degree six in the $\mathbf{a}_P$ either involves five or fewer of the matrices, is reducible, or is equivalent* to some linear combination of the invariants $\text{tr } \mathbf{a}_K \mathbf{a}_L \mathbf{a}_M \mathbf{a}_N \mathbf{a}_S \mathbf{a}_T$. These results are sharper than those obtained in [3], where it was shown by an argument based on Peano's theorem that every irreducible invariant of the $\mathbf{a}_P$ is of degree less than or equal to seven in the $\mathbf{a}_P$ and any invariant of degree seven either involves six or fewer of the matrices, is reducible or is equivalent to some linear combination of the invariants $\text{tr } \mathbf{a}_K \mathbf{a}_L \mathbf{a}_M \mathbf{a}_N \mathbf{a}_S \mathbf{a}_T \mathbf{a}_U$.

The problem of finding an integrity basis for any number of symmetric 3×3 matrices is thus reduced to that of finding an integrity basis for five symmetric 3×3 matrices.

10. Integrity basis for five symmetric 3×3 matrices

In a previous paper [3], finite integrity bases for five or fewer symmetric 3×3 matrices, under the orthogonal group of transformations, were derived. In [4], it was shown that certain elements of these bases are reducible, and equivalences were established between other elements of the bases. Integrity bases containing fewer elements were thus obtained. In particular, it was shown that an integrity basis for five symmetric 3×3 matrices $\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d}$ and $\mathbf{e}$ consists of the integrity bases for the five sets of four matrices which can be selected from $\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d}, \mathbf{e}$, together with the invariants

$$\begin{aligned} \text{(i)} \quad & \text{tr } \mathbf{a} \mathbf{b} \mathbf{c} \mathbf{d} \mathbf{e}, \quad \text{tr } \mathbf{a} \mathbf{b} \mathbf{d} \mathbf{e} \mathbf{c}, \quad \text{tr } \mathbf{a} \mathbf{b} \mathbf{e} \mathbf{c} \mathbf{d}, \\ & \text{tr } \mathbf{a} \mathbf{c} \mathbf{d} \mathbf{b} \mathbf{e}, \quad \text{tr } \mathbf{a} \mathbf{c} \mathbf{e} \mathbf{b} \mathbf{d}, \quad \text{tr } \mathbf{a} \mathbf{d} \mathbf{b} \mathbf{b} \mathbf{c} \mathbf{e}; \\ \text{(ii)} \quad & \text{tr } \mathbf{a} \mathbf{b} \mathbf{c} \mathbf{d} \mathbf{e}^2, \quad \text{tr } \mathbf{c} \mathbf{a} \mathbf{b} \mathbf{d} \mathbf{e}^2, \quad \text{tr } \mathbf{c} \mathbf{d} \mathbf{a} \mathbf{b} \mathbf{e}^2, \\ & \text{tr } \mathbf{a} \mathbf{c} \mathbf{d} \mathbf{b} \mathbf{e}^2, \quad \text{tr } \mathbf{d} \mathbf{a} \mathbf{c} \mathbf{b} \mathbf{e}^2, \quad \text{tr } \mathbf{a} \mathbf{d} \mathbf{b} \mathbf{c} \mathbf{e}^2 \end{aligned}$$

and invariants obtained from these by permuting $\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d}, \mathbf{e}$ cyclically;

$$\begin{aligned} \text{(iii)} \quad & \text{tr } \mathbf{a} \mathbf{b} \mathbf{e} \mathbf{c} \mathbf{d} \mathbf{e}^2, \quad \text{tr } \mathbf{b} \mathbf{c} \mathbf{e} \mathbf{a} \mathbf{d} \mathbf{e}^2, \quad \text{tr } \mathbf{c} \mathbf{a} \mathbf{e} \mathbf{b} \mathbf{d} \mathbf{e}^2, \\ & \text{tr } \mathbf{b} \mathbf{c} \mathbf{d} \mathbf{e} \mathbf{a} \mathbf{e}^2, \quad \text{tr } \mathbf{c} \mathbf{b} \mathbf{a} \mathbf{e} \mathbf{d} \mathbf{e}^2, \quad \text{tr } \mathbf{c} \mathbf{a} \mathbf{d} \mathbf{e} \mathbf{b} \mathbf{e}^2, \quad \text{tr } \mathbf{c} \mathbf{d} \mathbf{b} \mathbf{e} \mathbf{a} \mathbf{e}^2 \end{aligned}$$

and invariants obtained from these by permuting $\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d}, \mathbf{e}$ cyclically;

$$\begin{aligned} \text{(iv)} \quad & \text{tr } \mathbf{a} \mathbf{b} \mathbf{c} \mathbf{d}^2 \mathbf{e}^2, \quad \text{tr } \mathbf{a} \mathbf{b} \mathbf{d} \mathbf{c}^2 \mathbf{e}^2, \quad \text{tr } \mathbf{a} \mathbf{d} \mathbf{c} \mathbf{b}^2 \mathbf{e}^2, \quad \text{tr } \mathbf{d} \mathbf{b} \mathbf{c} \mathbf{a}^2 \mathbf{e}^2, \\ & \text{tr } \mathbf{a} \mathbf{b} \mathbf{e} \mathbf{c}^2 \mathbf{d}^2, \quad \text{tr } \mathbf{a} \mathbf{c} \mathbf{e} \mathbf{b}^2 \mathbf{d}^2, \quad \text{tr } \mathbf{c} \mathbf{b} \mathbf{e} \mathbf{a}^2 \mathbf{d}^2, \\ & \text{tr } \mathbf{a} \mathbf{d} \mathbf{e} \mathbf{b}^2 \mathbf{c}^2, \quad \text{tr } \mathbf{b} \mathbf{d} \mathbf{e} \mathbf{a}^2 \mathbf{c}^2, \quad \text{tr } \mathbf{c} \mathbf{d} \mathbf{e} \mathbf{a}^2 \mathbf{b}^2. \end{aligned}$$

In this section it will be shown that the number of elements in the integrity basis can be further reduced.

* An invariant I is said to be reducible, if it can be expressed as a polynomial in invariants of degree less than the degree of I and this property of I is denoted $I \equiv 0$. If I_1 and I_2 are invariants and $I_1 - I_2 \equiv 0$, then I_1 and I_2 are said to be equivalent, and this is denoted $I_1 \equiv I_2$.

The following results, proofs of which were given in [3], will be used. Let $\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{x}$ and $\mathbf{y}$ by 3×3 matrices, not equal to the unit matrix $\mathbf{I}$. Then

$$\text{tr } \mathbf{y}(\mathbf{a}\mathbf{b}\mathbf{c} + \mathbf{a}\mathbf{c}\mathbf{b} + \mathbf{b}\mathbf{a}\mathbf{c} + \mathbf{b}\mathbf{c}\mathbf{a} + \mathbf{c}\mathbf{a}\mathbf{b} + \mathbf{c}\mathbf{b}\mathbf{a}) \equiv 0, \quad (10.1)$$

and

$$\text{tr } \mathbf{y}\mathbf{a}^2\mathbf{x}\mathbf{b}^2 \equiv 0. \quad (10.2)$$

Also, if $\mathbf{a}, \mathbf{b}$ and $\mathbf{c}$ are symmetric 3×3 matrices

$$\text{tr } \mathbf{a}^2\mathbf{b}^2\mathbf{c}^2 \equiv 0. \quad (10.3)$$

Use will also be made of the following Lemmas, which are consequences of the definition of matrix multiplication:

Lemma 7. *The trace of a matrix product formed from 3×3 matrices is unaltered by cyclic permutation of the factors in the product.*

Lemma 8. *The trace of a matrix product formed from symmetric 3×3 matrices is unaltered if the order of the factors in the product is reversed.*

It follows from § 9 that the trace of any product formed from the five symmetric 3×3 matrices $\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d}, \mathbf{e}$ is reducible, if it has degree greater than six. Since each of the invariants (iii) and (iv) has degree seven, they are all reducible and may be omitted from the integrity basis.

We now consider the invariant

$$\text{tr } \mathbf{a}\mathbf{b}\mathbf{c}\mathbf{d}\mathbf{e}^2 \quad (10.4)$$

and invariants formed from it by permutation of $\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d}$ and $\mathbf{e}$. The set (ii) consists of selected invariants of this type. It was shown in [4] that any invariant of this type can be expressed in terms of those given in (ii). We shall now derive a number of equivalence relations between these invariants, which enables us to construct an integrity basis involving fewer invariants of the type (10.4) than are included in (ii).

First, suppose that $\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d}$ and $\mathbf{e}$ are symmetric 3×3 matrices and in (10.2) replace $\mathbf{y}, \mathbf{a}, \mathbf{x}$ and $\mathbf{b}$ by $\mathbf{a}, \mathbf{b} + \mathbf{c}, \mathbf{d}$ and $\mathbf{e}$ respectively, obtaining

$$\text{tr } \mathbf{a}(\mathbf{b}^2 + \mathbf{b}\mathbf{c} + \mathbf{c}\mathbf{b} + \mathbf{c}^2)\mathbf{d}\mathbf{e}^2 \equiv 0.$$

Since, by relations of the type (10.2), $\text{tr } \mathbf{a}\mathbf{b}^2\mathbf{d}\mathbf{e}^2 \equiv 0$, and $\text{tr } \mathbf{a}\mathbf{c}^2\mathbf{d}\mathbf{e}^2 \equiv 0$, this last relation may be written

$$\text{tr } \mathbf{a}(\mathbf{b}\mathbf{c} + \mathbf{c}\mathbf{b})\mathbf{d}\mathbf{e}^2 \equiv 0. \quad (10.5)$$

In (10.3), replace $\mathbf{a}, \mathbf{b}$ and $\mathbf{c}$ by $\mathbf{a} + \mathbf{b}, \mathbf{c} + \mathbf{d}$ and $\mathbf{e}$ respectively. Then

$$\text{tr } (\mathbf{a} + \mathbf{b})^2(\mathbf{c} + \mathbf{d})^2\mathbf{e}^2 \equiv 0.$$

Expanding the above relation, and inserting relations of the type (10.3) wherever possible, this gives

$$\text{tr } (\mathbf{a}\mathbf{b} + \mathbf{b}\mathbf{a})(\mathbf{c}\mathbf{d} + \mathbf{d}\mathbf{c})\mathbf{e}^2 \equiv 0. \quad (10.6)$$

Relations between invariants derived from (10.4) by permutation of $\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d}$ and $\mathbf{e}$ may also be obtained by making various substitutions in the relation (10.1). It will be shown that any relation obtained in this way can also be derived from relations of the types (10.5) and (10.6). For, setting $\mathbf{y} = \mathbf{d}\mathbf{e}^2$ in (10.1), and grouping the factors in the indicated manner, we obtain

$$\text{tr } \mathbf{d}\mathbf{e}^2\mathbf{a}(\mathbf{b}\mathbf{c} + \mathbf{c}\mathbf{b}) + \text{tr } \mathbf{d}\mathbf{e}^2\mathbf{b}(\mathbf{a}\mathbf{c} + \mathbf{c}\mathbf{a}) + \text{tr } \mathbf{d}\mathbf{e}^2\mathbf{c}(\mathbf{a}\mathbf{b} + \mathbf{b}\mathbf{a}) \equiv 0, \quad (10.7)$$

a relation which clearly can be obtained by forming the sum of three relations of the type (10.5), and applying Lemma 7. Further, replacing y, a, b and c in (10.1) by ab, c, d and e^2 respectively, the relation

$$\text{tr } ab(cde^2 + ce^2d + dce^2 + de^2c + e^2cd + e^2dc) \equiv 0, \quad (10.8)$$

is obtained. Using Lemmas 7 and 8, (10.8) may be written

$$\text{tr}(ab + ba)(cd + dc)e^2 + \text{tr } c(ab + ba)de^2 \equiv 0, \quad (10.9)$$

and this result may also be obtained by adding (10.6) to a relation of the type (10.5). It may be verified that any relation between invariants of the type (10.4) which can be obtained by making substitutions in (10.1) may, by use of Lemmas 7 and 8, be expressed in one or other of the forms (10.7) or (10.8), or a form obtained from (10.7) or (10.8) by permuting the matrices a, b, c, d and e . Hence the relation (10.1) can yield no equivalences between invariants of the type (10.4) which are not contained in relations of the types (10.5) and (10.6). In considering equivalences between invariants of the form (10.4), therefore, only (10.5) and (10.6) and relations obtained from them by permuting the matrices will be used.

There are six distinct relations of the type (10.5) which are obtained by permuting a, b, c and d in (10.5). They are

$$\begin{aligned} \text{tr } abcde^2 + \text{tr } acbde^2 &\equiv 0, & \text{tr } abdc e^2 + \text{tr } adbce^2 &\equiv 0, \\ \text{tr } acdb e^2 + \text{tr } adcbe^2 &\equiv 0, & \text{tr } cabde^2 + \text{tr } cbade^2 &\equiv 0, \\ \text{tr } bacde^2 + \text{tr } bcade^2 &\equiv 0, & \text{tr } badce^2 + \text{tr } bdace^2 &\equiv 0. \end{aligned} \quad (10.10)$$

Also, there are three distinct relations of the type (10.6) which are obtained by permuting a, b, c and d in (10.6). These are

$$\text{tr } abcde^2 + \text{tr } abdc e^2 + \text{tr } bacde^2 + \text{tr } badce^2 \equiv 0, \quad (10.11)$$

$$\text{tr } acbde^2 + \text{tr } acdb e^2 + \text{tr } cabde^2 + \text{tr } bdace^2 \equiv 0, \quad (10.12)$$

$$\text{tr } adbce^2 + \text{tr } adcbe^2 + \text{tr } cbade^2 + \text{tr } bcade^2 \equiv 0. \quad (10.13)$$

In (10.12) and (10.13) the order of the factors in some of the invariants has been altered in accordance with Lemmas 7 and 8.

It is seen that the sum of the terms on the left of the six relations (10.10) is equal to the sum of the terms on the left of the three relations (10.11), (10.12), (10.13). Hence, not all of the nine equations (10.10) to (10.13) are independent, and one of them, say (10.13), can be regarded as a consequence of the remaining eight equations, and need not be considered further. By considering the matrix of the coefficients of the twelve invariants occurring in the eight relations (10.10), (10.11) and (10.12), it is found that these equations are independent, and may be used to express eight of the invariants, suitably chosen, in terms of the other four invariants. For example, (10.10), (10.11) and (10.12) may be used to express $\text{tr } acbde^2$, $\text{tr } adbce^2$, $\text{tr } acdb e^2$, $\text{tr } cabde^2$, $\text{tr } cbade^2$, $\text{tr } bcade^2$, $\text{tr } badce^2$ and $\text{tr } bdace^2$ in terms of $\text{tr } abcde^2$, $\text{tr } abdc e^2$, $\text{tr } adcbe^2$ and $\text{tr } bacde^2$. Hence of the invariant (10.4) and invariants derived from (10.4) by permuting a, b, c and d , it is necessary to retain in the integrity basis only these last four invariants. Therefore, of the invariant (10.4) and invariants derived from (10.4) by permutation of the matrices a, b, c, d and e , it is sufficient to retain in the

integrality basis only the invariants

$$(ii') \quad \text{tr } \mathbf{a} \mathbf{b} \mathbf{c} \mathbf{d} \mathbf{e}^2, \quad \text{tr } \mathbf{a} \mathbf{b} \mathbf{d} \mathbf{c} \mathbf{e}^2, \quad \text{tr } \mathbf{a} \mathbf{d} \mathbf{c} \mathbf{b} \mathbf{e}^2, \quad \text{tr } \mathbf{b} \mathbf{a} \mathbf{c} \mathbf{d} \mathbf{e}^2$$

and invariants obtained from these by permuting the matrices $\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d}$ and $\mathbf{e}$ cyclically. We thus have the conclusion that an integrality basis for the five symmetric 3×3 matrices $\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d}$ and $\mathbf{e}$ may be formed by the integrality bases for the five sets of four matrices which can be selected from $\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d}$ and $\mathbf{e}$, together with the invariants enumerated in (i) and (ii') and, invariants obtained from (ii') by permuting $\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d}$ and $\mathbf{e}$ cyclically.

11. Application to the mechanics of materials with memory

In a previous paper [6], GREEN, RIVLIN & SPENCER have discussed the form taken by the constitutive equation for deformations of an isotropic material if the stress components in an element of the material at the instant of time t , referred to a rectangular Cartesian coordinate system x , are assumed to have continuous functional dependence on the deformation gradients in it at all times up to the time t and particular dependence on the deformation gradients at the instant t . The deformation is described by the dependence of the coordinates $x_i(\tau)$ of a generic particle of the body on the time τ and its coordinates X_i when $\tau = 0$.

The following notation is employed:

$$\begin{aligned} x_i(t) &= x_i, & g_{ij}(\tau) &= \frac{\partial x_k(\tau)}{\partial X_i} \frac{\partial x_k(\tau)}{\partial X_j}, \\ \mathbf{g}(\tau) &= \|g_{ij}(\tau)\|, & \mathbf{g} &= \mathbf{g}(t), & \mathbf{c} &= \|\partial x_i / \partial X_j\|, \\ & & \mathbf{c}' &= \text{transpose of } \mathbf{c}. \end{aligned} \quad (11.1)$$

With this notation it was shown that the stress matrix $\mathbf{T}$ in the coordinate system x is given by

$$\mathbf{T} = \mathbf{c} \mathbf{F} \mathbf{c}', \quad (11.2)$$

where

$$\mathbf{F} = \sum_{\nu} \vartheta_{\nu} \mathbf{X}_{\nu}^{(0)} + \sum_{\mu=1}^6 \sum_{\nu} \int_0^t \int_0^t \cdots \int_0^t \vartheta_{\nu}(\tau_1, \tau_2, \dots, \tau_{\mu}) (\mathbf{X}_{\nu}^{(\mu)} + \mathbf{X}_{\nu}^{(\mu)'}) d\tau_1 d\tau_2 \dots d\tau_{\mu}. \quad (11.3)$$

In (11.3), $\mathbf{X}_{\nu}^{(\mu)}$ for the various values of ν are the matrix products in Table 1 which have all of the matrices $\mathbf{g}(\tau_1), \mathbf{g}(\tau_2), \dots, \mathbf{g}(\tau_{\mu})$ as factors. $\mathbf{X}_{\nu}^{(\mu)'}$ is the transpose of $\mathbf{X}_{\nu}^{(\mu)}$. $\vartheta_{\nu}(\tau_1, \tau_2, \dots, \tau_{\mu})$ is a function of $t, \tau_1, \tau_2, \dots, \tau_{\mu}$ and a polynomial in the invariants

$$\text{tr } \tilde{\omega}_{\alpha}^{(0)} \text{ and } \int_0^t \int_0^t \cdots \int_0^t \varphi(\tau_1, \tau_2, \dots, \tau_{\beta}) \text{tr } \tilde{\omega}_{\alpha}^{(\beta)} d\tau_1 d\tau_2 \dots d\tau_{\beta}, \quad (\beta=1, 2, \dots, 7) \quad (11.4)$$

in which $\text{tr } \tilde{\omega}_{\alpha}^{(0)}$ are the invariants in Table 2 which involve none of the matrices $\mathbf{g}(\tau_{\beta})$ ($\beta=1, 2, \dots, 7$) and $\text{tr } \tilde{\omega}_{\alpha}^{(\beta)}$ are the invariants in Table 2 which involve all of the matrices $\mathbf{g}(\tau_1), \mathbf{g}(\tau_2), \dots, \mathbf{g}(\tau_{\beta})$.

By an argument similar to that employed in deriving this result and using the results developed in previous sections of the present paper, it follows that the stress matrix $\mathbf{T}$ may be expressed in the form (11.2), where $\mathbf{F}$ is given by

$$\mathbf{F} = \sum_{\nu} \vartheta_{\nu} \mathbf{X}_{\nu}^{(0)} + \sum_{\mu=1}^5 \sum_{\nu} \int_0^t \int_0^t \cdots \int_0^t \vartheta_{\nu}(\tau_1, \tau_2, \dots, \tau_{\mu}) (\mathbf{X}_{\nu}^{(\mu)} + \mathbf{X}_{\nu}^{(\mu)'}) d\tau_1 d\tau_2 \dots d\tau_{\mu}. \quad (11.5)$$

$\chi_v^{(\mu)}$ are now the matrix products above the line in Table 1 which have all of the matrices $\mathbf{g}(\tau_1), \mathbf{g}(\tau_2), \dots, \mathbf{g}(\tau_\mu)$ as factors. $\partial_v(\tau_1, \tau_2, \dots, \tau_\mu)$ is a function of t , $\tau_1, \tau_2, \dots, \tau_\mu$ and a polynomial in the invariants (11.4) with $\beta = 1, 2, \dots, 6$. We note that $\text{tr } \tilde{\omega}_\alpha^{(\beta)}$ ($\beta = 0, 1, \dots, 6$) are now the invariants above the line in Table 2.

Table 1

$$\begin{aligned}
 & \mathbf{I}; \\
 & \mathbf{g}, \mathbf{g}^2; \\
 & \mathbf{g}\mathbf{g}(\tau_1), \quad \mathbf{g}^2\mathbf{g}(\tau_1); \\
 & \mathbf{g}\mathbf{g}(\tau_1)\mathbf{g}(\tau_2), \quad \mathbf{g}(\tau_1)\mathbf{g}\mathbf{g}(\tau_2), \quad \mathbf{g}^2\mathbf{g}(\tau_1)\mathbf{g}(\tau_2), \quad \mathbf{g}(\tau_1)\mathbf{g}^2\mathbf{g}(\tau_2), \\
 & \mathbf{g}\mathbf{g}(\tau_1)\mathbf{g}(\tau_2)\mathbf{g}^2; \\
 & \mathbf{g}\mathbf{g}(\tau_1)\mathbf{g}(\tau_2)\mathbf{g}(\tau_3), \quad \mathbf{g}(\tau_1)\mathbf{g}\mathbf{g}(\tau_2)\mathbf{g}(\tau_3), \quad \mathbf{g}^2\mathbf{g}(\tau_1)\mathbf{g}(\tau_2)\mathbf{g}(\tau_3), \\
 & \mathbf{g}(\tau_1)\mathbf{g}^2\mathbf{g}(\tau_2)\mathbf{g}(\tau_3); \\
 & \mathbf{g}\mathbf{g}(\tau_1)\mathbf{g}(\tau_2)\mathbf{g}(\tau_3)\mathbf{g}(\tau_4), \quad \mathbf{g}(\tau_1)\mathbf{g}\mathbf{g}(\tau_2)\mathbf{g}(\tau_3)\mathbf{g}(\tau_4), \\
 & \mathbf{g}(\tau_1)\mathbf{g}(\tau_2)\mathbf{g}\mathbf{g}(\tau_3)\mathbf{g}(\tau_4); \\
 & \mathbf{g}(\tau_1), \quad \mathbf{g}(\tau_1)\mathbf{g}(\tau_2), \quad \mathbf{g}(\tau_1)\mathbf{g}(\tau_2)\mathbf{g}(\tau_3), \quad \mathbf{g}(\tau_1)\mathbf{g}(\tau_2)\mathbf{g}(\tau_3)\mathbf{g}(\tau_4), \\
 & \mathbf{g}(\tau_1)\mathbf{g}(\tau_2)\mathbf{g}(\tau_3)\mathbf{g}(\tau_4)\mathbf{g}(\tau_5); \\
 & \mathbf{g}\mathbf{g}(\tau_1)\mathbf{g}^2\mathbf{g}(\tau_2); \\
 & \mathbf{g}\mathbf{g}(\tau_1)\mathbf{g}(\tau_2)\mathbf{g}(\tau_3)\mathbf{g}^2, \quad \mathbf{g}(\tau_1)\mathbf{g}\mathbf{g}(\tau_2)\mathbf{g}(\tau_3)\mathbf{g}^2, \\
 & \mathbf{g}(\tau_1)\mathbf{g}(\tau_2)\mathbf{g}\mathbf{g}(\tau_3)\mathbf{g}^2, \quad \mathbf{g}(\tau_1)\mathbf{g}\mathbf{g}(\tau_2)\mathbf{g}^2\mathbf{g}(\tau_3); \\
 & \mathbf{g}^2\mathbf{g}(\tau_1)\mathbf{g}(\tau_2)\mathbf{g}(\tau_3)\mathbf{g}(\tau_4), \quad \mathbf{g}(\tau_1)\mathbf{g}^2\mathbf{g}(\tau_2)\mathbf{g}(\tau_3)\mathbf{g}(\tau_4), \\
 & \mathbf{g}(\tau_1)\mathbf{g}(\tau_2)\mathbf{g}^2\mathbf{g}(\tau_3)\mathbf{g}(\tau_4); \\
 & \mathbf{g}\mathbf{g}(\tau_1)\mathbf{g}(\tau_2)\mathbf{g}(\tau_3)\mathbf{g}(\tau_4)\mathbf{g}(\tau_5), \quad \mathbf{g}(\tau_1)\mathbf{g}\mathbf{g}(\tau_2)\mathbf{g}(\tau_3)\mathbf{g}(\tau_4)\mathbf{g}(\tau_5), \\
 & \mathbf{g}(\tau_1)\mathbf{g}(\tau_2)\mathbf{g}\mathbf{g}(\tau_3)\mathbf{g}(\tau_4)\mathbf{g}(\tau_5); \\
 & \mathbf{g}(\tau_1)\mathbf{g}(\tau_2)\mathbf{g}(\tau_3)\mathbf{g}(\tau_4)\mathbf{g}(\tau_5)\mathbf{g}(\tau_6).
 \end{aligned}$$

Table 2

$$\begin{aligned}
 & \text{tr } \mathbf{g}, \quad \text{tr } \mathbf{g}^2, \quad \text{tr } \mathbf{g}^3; \\
 & \text{tr } \mathbf{g}\mathbf{g}(\tau_1), \quad \text{tr } \mathbf{g}^2\mathbf{g}(\tau_1); \\
 & \text{tr } \mathbf{g}\mathbf{g}(\tau_1)\mathbf{g}(\tau_2), \quad \text{tr } \mathbf{g}^2\mathbf{g}(\tau_1)\mathbf{g}(\tau_2); \\
 & \text{tr } \mathbf{g}\mathbf{g}(\tau_1)\mathbf{g}(\tau_2)\mathbf{g}(\tau_3), \quad \text{tr } \mathbf{g}^2\mathbf{g}(\tau_1)\mathbf{g}(\tau_2)\mathbf{g}(\tau_3); \\
 & \text{tr } \mathbf{g}(\tau_1)\mathbf{g}(\tau_2)\mathbf{g}\mathbf{g}(\tau_3)\mathbf{g}^2; \\
 & \text{tr } \mathbf{g}\mathbf{g}(\tau_1)\mathbf{g}(\tau_2)\mathbf{g}(\tau_3)\mathbf{g}(\tau_4), \quad \text{tr } \mathbf{g}^2\mathbf{g}(\tau_1)\mathbf{g}(\tau_2)\mathbf{g}(\tau_3)\mathbf{g}(\tau_4); \\
 & \text{tr } \mathbf{g}\mathbf{g}(\tau_1)\mathbf{g}(\tau_2)\mathbf{g}(\tau_3)\mathbf{g}(\tau_4)\mathbf{g}(\tau_5); \\
 & \text{tr } \mathbf{g}(\tau_1), \quad \text{tr } \mathbf{g}(\tau_1)\mathbf{g}(\tau_2), \quad \text{tr } \mathbf{g}(\tau_1)\mathbf{g}(\tau_2)\mathbf{g}(\tau_3), \\
 & \text{tr } \mathbf{g}(\tau_1)\mathbf{g}(\tau_2)\mathbf{g}(\tau_3)\mathbf{g}(\tau_4), \quad \text{tr } \mathbf{g}(\tau_1)\mathbf{g}(\tau_2)\mathbf{g}(\tau_3)\mathbf{g}(\tau_4)\mathbf{g}(\tau_5), \\
 & \text{tr } \mathbf{g}(\tau_1)\mathbf{g}(\tau_2)\mathbf{g}(\tau_3)\mathbf{g}(\tau_4)\mathbf{g}(\tau_5)\mathbf{g}(\tau_6); \\
 & \text{tr } \mathbf{g}(\tau_1)\mathbf{g}\mathbf{g}(\tau_2)\mathbf{g}(\tau_3)\mathbf{g}(\tau_4)\mathbf{g}^2, \quad \text{tr } \mathbf{g}(\tau_1)\mathbf{g}(\tau_2)\mathbf{g}\mathbf{g}(\tau_3)\mathbf{g}(\tau_4)\mathbf{g}^2, \\
 & \text{tr } \mathbf{g}(\tau_1)\mathbf{g}(\tau_2)\mathbf{g}(\tau_3)\mathbf{g}\mathbf{g}(\tau_4)\mathbf{g}^2; \\
 & \text{tr } \mathbf{g}^2\mathbf{g}(\tau_1)\mathbf{g}(\tau_2)\mathbf{g}(\tau_3)\mathbf{g}(\tau_4)\mathbf{g}(\tau_5); \\
 & \text{tr } \mathbf{g}\mathbf{g}(\tau_1)\mathbf{g}(\tau_2)\mathbf{g}(\tau_3)\mathbf{g}(\tau_4)\mathbf{g}(\tau_5)\mathbf{g}(\tau_6); \\
 & \text{tr } \mathbf{g}(\tau_1)\mathbf{g}(\tau_2)\mathbf{g}(\tau_3)\mathbf{g}(\tau_4)\mathbf{g}(\tau_5)\mathbf{g}(\tau_6)\mathbf{g}(\tau_7).
 \end{aligned}$$

12. Appendix. An alternative derivation of Lemma 6

Lemma 6, which expresses the fact that any matrix product of degree six in the $\mathbf{a}_P$ may be expressed as a matrix polynomial of degree five or less in the $\mathbf{a}_P$, may also be derived in the following way. It was shown in [3] that any invariant of the form $\text{tr } \mathbf{a}_S^2 \mathbf{a}_R \mathbf{a}_M^2 \mathbf{y}$ is reducible; that is, in the notation of [3]

and [4]

$$\text{tr } \mathbf{a}_S^2 \mathbf{a}_K \mathbf{a}_M^2 \mathbf{y} \equiv 0, \quad (\mathbf{y} \neq \mathbf{I}). \quad (12.1)$$

Replacing $\mathbf{a}_M$ by $\mathbf{a}_M + \mathbf{a}_N$, and using the relations $\text{tr } \mathbf{a}_S^2 \mathbf{a}_K \mathbf{a}_M^2 \mathbf{y} \equiv 0$, $\text{tr } \mathbf{a}_S^2 \mathbf{a}_K \mathbf{a}_N^2 \mathbf{y} \equiv 0$, this becomes

$$\text{tr } \mathbf{a}_S^2 \mathbf{a}_K (\mathbf{a}_M \mathbf{a}_N + \mathbf{a}_N \mathbf{a}_M) \mathbf{y} \equiv 0. \quad (12.2)$$

Now in (12.2) replace $\mathbf{a}_K$ by $\mathbf{a}_K \mathbf{a}_L$. This gives

$$\text{tr } \mathbf{a}_S^2 \mathbf{a}_K \mathbf{a}_L (\mathbf{a}_M \mathbf{a}_N + \mathbf{a}_N \mathbf{a}_M) \mathbf{y} \equiv 0. \quad (12.3)$$

Next, in (12.2), replace $\mathbf{a}_M$ by $\mathbf{a}_L \mathbf{a}_M$

$$\text{tr } \mathbf{a}_S^2 \mathbf{a}_K (\mathbf{a}_L \mathbf{a}_M \mathbf{a}_N + \mathbf{a}_N \mathbf{a}_L \mathbf{a}_M) \mathbf{y} \equiv 0. \quad (12.4)$$

Also in (12.2), replace $\mathbf{a}_M$ by $\mathbf{a}_L$, and $\mathbf{y}$ by $\mathbf{a}_M \mathbf{y}$

$$\text{tr } \mathbf{a}_S^2 \mathbf{a}_K (\mathbf{a}_L \mathbf{a}_N + \mathbf{a}_N \mathbf{a}_L) \mathbf{a}_M \mathbf{y} \equiv 0. \quad (12.5)$$

Adding (12.3) to (12.4), and subtracting (12.5), it follows that

$$\text{tr } \mathbf{a}_S^2 \mathbf{a}_K \mathbf{a}_L \mathbf{a}_M \mathbf{a}_N \mathbf{y} \equiv 0. \quad (12.6)$$

Since the trace of a matrix product is unaltered by a cyclic permutation of the factors of the product, it follows by making various substitutions in (12.6) that

$$\begin{aligned} \text{tr } \mathbf{a}_K \mathbf{a}_S^2 \mathbf{a}_L \mathbf{a}_M \mathbf{a}_N \mathbf{y} &\equiv 0, & \text{tr } \mathbf{a}_K \mathbf{a}_L \mathbf{a}_S^2 \mathbf{a}_M \mathbf{a}_N \mathbf{y} &\equiv 0, \\ \text{tr } \mathbf{a}_K \mathbf{a}_L \mathbf{a}_M \mathbf{a}_S^2 \mathbf{a}_N \mathbf{y} &\equiv 0, & \text{tr } \mathbf{a}_K \mathbf{a}_L \mathbf{a}_M \mathbf{a}_N \mathbf{a}_S^2 \mathbf{y} &\equiv 0. \end{aligned} \quad (12.7)$$

In for example (12.6), replace $\mathbf{a}_S$ by $\mathbf{a}_S + \mathbf{a}_T$, and insert the relations $\text{tr } \mathbf{a}_S^2 \mathbf{a}_K \mathbf{a}_L \mathbf{a}_M \mathbf{a}_N \mathbf{y} \equiv 0$, $\text{tr } \mathbf{a}_T^2 \mathbf{a}_K \mathbf{a}_L \mathbf{a}_M \mathbf{a}_N \mathbf{y} \equiv 0$. This gives

$$\text{tr } (\mathbf{a}_S \mathbf{a}_T + \mathbf{a}_T \mathbf{a}_S) \mathbf{a}_K \mathbf{a}_L \mathbf{a}_M \mathbf{a}_N \mathbf{y} \equiv 0. \quad (12.8)$$

In a similar way, each of the relations (12.7) leads to a relation in which $\mathbf{a}_S^2$ in the original relation is replaced by $\mathbf{a}_S \mathbf{a}_T + \mathbf{a}_T \mathbf{a}_S$. It follows that any invariant of the form $\text{tr } \mathbf{a}_K \mathbf{a}_L \mathbf{a}_M \mathbf{a}_N \mathbf{a}_S \mathbf{a}_T \mathbf{y}$ is equivalent to the negative of any invariant formed from $\text{tr } \mathbf{a}_K \mathbf{a}_L \mathbf{a}_M \mathbf{a}_N \mathbf{a}_S \mathbf{a}_T \mathbf{y}$ by the interchange of any two adjacent factors from $\mathbf{a}_K, \mathbf{a}_L, \mathbf{a}_M, \mathbf{a}_N, \mathbf{a}_S$ and $\mathbf{a}_T$. Hence any invariant of the form $\text{tr } \mathbf{a}_K \mathbf{a}_L \mathbf{a}_M \mathbf{a}_N \mathbf{a}_S \mathbf{a}_T \mathbf{y}$ is equivalent to any invariant formed from it by an even permutation of the factors $\mathbf{a}_K, \mathbf{a}_L, \mathbf{a}_M, \mathbf{a}_N, \mathbf{a}_S$ and $\mathbf{a}_T$, and to the negative of any invariant formed from it by an odd permutation of these factors.

It was also shown in [3] that if $\mathbf{a}, \mathbf{b}, \mathbf{c}$ and $\mathbf{y}$ are 3×3 matrices, not equal to $\mathbf{I}$, then

$$\text{tr } (\mathbf{a} \mathbf{b} \mathbf{c} + \mathbf{a} \mathbf{c} \mathbf{b} + \mathbf{b} \mathbf{a} \mathbf{c} + \mathbf{b} \mathbf{c} \mathbf{a} + \mathbf{c} \mathbf{a} \mathbf{b} + \mathbf{c} \mathbf{b} \mathbf{a}) \mathbf{y} \equiv 0. \quad (12.9)$$

In (12.9), replace $\mathbf{a}, \mathbf{b}$ and $\mathbf{c}$ by $\mathbf{a}_K \mathbf{a}_L, \mathbf{a}_M \mathbf{a}_N$ and $\mathbf{a}_S \mathbf{a}_T$ respectively, obtaining

$$\begin{aligned} \text{tr } (\mathbf{a}_K \mathbf{a}_L \mathbf{a}_M \mathbf{a}_N \mathbf{a}_S \mathbf{a}_T + \mathbf{a}_K \mathbf{a}_L \mathbf{a}_S \mathbf{a}_T \mathbf{a}_M \mathbf{a}_N + \mathbf{a}_M \mathbf{a}_N \mathbf{a}_K \mathbf{a}_L \mathbf{a}_S \mathbf{a}_T + \\ + \mathbf{a}_M \mathbf{a}_N \mathbf{a}_S \mathbf{a}_T \mathbf{a}_K \mathbf{a}_L + \mathbf{a}_S \mathbf{a}_T \mathbf{a}_K \mathbf{a}_L \mathbf{a}_M \mathbf{a}_N + \mathbf{a}_S \mathbf{a}_T \mathbf{a}_M \mathbf{a}_N \mathbf{a}_K \mathbf{a}_L) \mathbf{y} \equiv 0. \end{aligned} \quad (12.10)$$

It may be verified that each of the invariants in (12.10) may be derived from $\text{tr } \mathbf{a}_K \mathbf{a}_L \mathbf{a}_M \mathbf{a}_N \mathbf{a}_S \mathbf{a}_T \mathbf{y}$ by an even permutation of $\mathbf{a}_K, \mathbf{a}_L, \mathbf{a}_M, \mathbf{a}_N, \mathbf{a}_S$ and $\mathbf{a}_T$. Hence each of these invariants is equivalent to $\text{tr } \mathbf{a}_K \mathbf{a}_L \mathbf{a}_M \mathbf{a}_N \mathbf{a}_S \mathbf{a}_T \mathbf{y}$, and it follows from (12.10) that

$$\text{tr } \mathbf{a}_K \mathbf{a}_L \mathbf{a}_M \mathbf{a}_N \mathbf{a}_S \mathbf{a}_T \mathbf{y} \equiv 0. \quad (12.11)$$

Now denote the invariant $\text{tr } \mathbf{a}_K \mathbf{a}_L \mathbf{a}_M \mathbf{a}_N \mathbf{a}_S \mathbf{a}_T \mathbf{y}$ by J . By (12.11), J is a reducible invariant, and so can be expressed as a polynomial in invariants of lower degree. From the manner in which (12.11) is derived, it follows that the expression for J must be of the form

$$J = \sum_{\alpha} \varphi_{\alpha} J_{\alpha}, \quad (12.12)$$

where the φ_{α} are invariants which are polynomials in traces of matrix products of the matrices $\mathbf{a}_K, \mathbf{a}_L, \mathbf{a}_M, \mathbf{a}_N, \mathbf{a}_S$ and $\mathbf{a}_T$, each of the φ_{α} being of total degree less than or equal to six in the elements of these matrices; and the J_{α} are invariants of the form $\text{tr } \mathbf{\Pi}_{\alpha} \mathbf{y}$, where each $\mathbf{\Pi}_{\alpha}$ is a matrix product formed from five or fewer of the factors $\mathbf{a}_K, \mathbf{a}_L, \mathbf{a}_M, \mathbf{a}_N, \mathbf{a}_S$ and $\mathbf{a}_T$.

The matrix product $\mathbf{a}_K \mathbf{a}_L \mathbf{a}_M \mathbf{a}_N \mathbf{a}_S \mathbf{a}_T$ is now denoted by $\mathbf{\Pi}$, and the ij^{th} elements of $\mathbf{\Pi}$ and $\mathbf{y}$ are denoted by Π_{ij} and y_{ij} respectively. Then, from (12.11), (12.12), and the definition of the trace of a matrix

$$\begin{aligned} \Pi_{ij} &= \frac{\partial}{\partial y_{ji}} (\text{tr } \mathbf{\Pi} \mathbf{y}), \\ &= \partial J / \partial y_{ji}, \\ &= \sum_{\alpha} \varphi_{\alpha} \partial J_{\alpha} / \partial y_{ji}. \end{aligned}$$

But, from the definition of the J_{α} , $\partial J_{\alpha} / \partial y_{ji}$ is the ij^{th} element of $\mathbf{\Pi}_{\alpha}$. Hence

$$\mathbf{\Pi} = \sum_{\alpha} \varphi_{\alpha} \mathbf{\Pi}_{\alpha}.$$

Since the $\mathbf{\Pi}_{\alpha}$ are all matrix products of degree less than or equal to five in the $\mathbf{a}_P$, the last relation may be written

$$\mathbf{\Pi} = \mathbf{Q}(5),$$

which is an expression of Lemma 6.

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References

- [1] RIVLIN, R. S., & J. L. ERICKSEN: J. Rational Mech. Anal. **4**, 323 (1955).
- [2] RIVLIN, R. S.: J. Rational Mech. Anal. **4**, 681 (1955).
- [3] SPENCER, A. J. M., & R. S. RIVLIN: Arch. Rational Mech. Anal. **2**, 309 (1959).
- [4] SPENCER, A. J. M., & R. S. RIVLIN: Arch. Rational Mech. Anal. **2**, 435 (1959).
- [5] ADKINS, J. E.: Phil. Trans. Roy. Soc. A **250**, 519 (1958).
- [6] GREEN, A. E., R. S. RIVLIN & A. J. M. SPENCER: Arch. Rational Mech. Anal. **3**, 82 (1959).

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Anisotropic Fluids

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1. Introduction

Elsewhere [1], the author formulated what may be the simplest properly invariant theory of anisotropic fluids. An analysis of simple shearing motion given there suggested that this type of theory might be applicable to some fluids which are commonly treated as isotropic and might describe them more accurately than comparably simple theories of isotropic fluids. The purpose of this paper is to present a revised theory which seems to us better adapted for exploring this possibility. This theory incorporates some features of the theory proposed by OSEEN [2] to describe the behavior of the so-called liquid crystals. However, it excludes a type of long range interaction incorporated in his theory. It seems to be necessary to include some such complication to obtain an adequate description of these rather peculiar fluids. We prefer to avoid this, at least until the simpler theory presented here is better understood.

Recently, GREEN & RIVLIN [3], NOLL [4] and GREEN, RIVLIN & SPENCER [5] have formulated very general theories of materials. Existing frameworks of this type seem not to be sufficiently flexible to properly include the theory presented here.

Cartesian tensor notation is used throughout.

2. Conservation Laws

Let V be any material volume with boundary S and let da_i denote the vector element of area on S , directed outward relative to V . We assume the basic laws of conservation or balance of mass, linear momentum, moment of momentum and energy, respectively, in the forms

$$(2.1) \quad \frac{d}{dt} \int_V \rho dV = 0,$$

$$(2.2) \quad \frac{d}{dt} \int_V p_i dV = \oint_S t_{ij} da_j + \int_V f_i dV,$$

$$(2.3) \quad \frac{d}{dt} \int_V m_{ij} dV = \oint_S x_{[i} t_{j]k} da_k + \int_V x_{[i} f_{j]} dV,$$

$$(2.4) \quad \frac{d}{dt} \int_V E dV = \oint_S t_{ij} \dot{x}_i da_j + \int_V f_i \dot{x}_i dV - \oint_S q_i da_i.$$

Here ρ , p_i , $m_{ij} = -m_{ji}$ and E are the volume densities of total mass, linear momentum, bivector moment of momentum and energy, respectively; t_i is the

stress tensor, f_i the body force per unit volume and q_i the heat flux. The square bracket means "the antisymmetric part of", *e.g.*

$$x_{[i}f_{j]} = \frac{1}{2}(x_i f_j - x_j f_i).$$

We envisage a fluid having at each particle a single preferred direction, which we represent by a vector n_i of variable magnitude. One might picture it as a fluid composed of dumbbell molecules. Pursuing this analogy, we note that the total linear momentum, moment of momentum and kinetic energy, respectively, of a pair of mass points can be written in the form

$$M \dot{x}_i, \quad M x_{[i} \dot{x}_{j]} + M n_{[i} \dot{n}_{j]}, \quad \frac{1}{2} M (\dot{x}_i \dot{x}_i + \dot{n}_i \dot{n}_i).$$

Here

$$M = M_1 + M_2, \quad M x_i = M_1 y_i + M_2 z_i, \quad M n_i = \sqrt{M_1 M_2} (y_i - z_i),$$

M_1 and M_2 being the two masses, located at points with coordinates y_i and z_i , respectively. This serves to motivate writing

$$(2.5) \quad \phi_i = \varrho \dot{x}_i,$$

$$(2.6) \quad m_{ij} = x_{[i} \phi_{j]} + \varrho n_{[i} \dot{n}_{j]},$$

$$(2.7) \quad E = \varrho \varepsilon + \frac{1}{2} \varrho (\dot{x}_i \dot{x}_i + \dot{n}_i \dot{n}_i),$$

where ε is the total less kinetic energy density, *i.e.*, the internal energy density. Equations (2.5)–(2.7) may of course be useful in describing quite different types of structure.

Following OSEEN [2], we introduce one more equation of the form

$$(2.8) \quad \frac{d}{dt} \int_V \varrho \dot{n}_i dV = \int_V g_i dV.$$

He writes g_i as the sum of an extrinsic part, which he suggests might be non-zero if there are applied magnetic fields, and an intrinsic part. In constructing the theory described here, we have assumed the extrinsic part vanishes.

For sufficiently smooth functions, it is a straightforward matter to derive differential forms of these relations using the divergence theorem and the identity

$$\frac{d}{dt} \int_V f dV = \int_V (\dot{f} + f \dot{x}_{i,i}) dV.$$

After some simplification, one obtains

$$(2.9) \quad \dot{\varrho} + \varrho \dot{x}_{i,i} = 0,$$

$$(2.10) \quad \varrho \ddot{x}_i = t_{ij,j} + f_i,$$

$$(2.11) \quad \varrho \ddot{n}_i = g_i,$$

$$(2.12) \quad \varrho \dot{\varepsilon} = t_{ij} \dot{x}_{i,j} - \varrho \dot{n}_i \ddot{n}_i - q_{i,i},$$

$$(2.13) \quad n_{[i} g_{j]} = t_{[ij]}.$$

If we interpret OSEEN correctly, his theory is consistent with (2.9)–(2.12), but not with (2.13). He assumes that $t_{ij} = t_{ji}$, as is customary in mechanics. However, there is some precedent for (2.13). In his theory of elastic dielectrics TOUPIN

[6, eqn. (9.10)] introduced an equation formally identical with (2.13), with n_i interpreted as the polarization vector, g_i interpreted as an intrinsic local part of the electric field and t_{ij} as a local part of the stress tensor. GRAD [7, § 4] discusses conservation laws more general than those used here. In doing so, he introduces an equation similar to (2.13).

Let

$$(2.14) \quad d_{ij} \equiv \frac{1}{2}(\dot{x}_{i,j} + \dot{x}_{j,i}), \quad w_{ij} \equiv \dot{x}_{[i,j]}, \quad \dot{x}_{i,j} = d_{ij} + w_{ij},$$

$$(2.15) \quad \hat{n}_i = \dot{n}_i - w_{ij}n_j.$$

The vector $\hat{n}_i$ plays an important role in the theory discussed by ERICKSEN [1]. It transforms as a vector under time dependent orthogonal coordinate transformations, whereas $\dot{n}_i$ does not. Using (2.13), (2.14) and (2.15), we obtain

$$(2.16) \quad \begin{aligned} t_{ij}\dot{x}_{i,j} &= t_{ij}d_{ij} + t_{[ij]}w_{ij} \\ &= t_{ij}d_{ij} + n_{[j}g_{i]}w_{ij} \\ &= t_{ij}d_{ij} + n_jg_iw_{ij} \\ &= t_{ij}d_{ij} + g_i(\dot{n}_i - \hat{n}_i). \end{aligned}$$

Using (2.16), we can rewrite the energy equation (2.12) in the form

$$(2.17) \quad \rho \dot{E} = t_{ij}d_{ij} - g_i\hat{n}_i - q_{i,i}.$$

3. Constitutive Equations

As a general principle to be used in constructing constitutive equations, TRUESDELL [8, § 19] has proposed that the stress and heat flux should be assumed to depend on the same variables, calling this BRILLOUIN'S *principle*. We adopt an extension of this, assuming that t_{ij} , q_i and g_i at a particle P at time t are all functions of the variables

$$(3.1) \quad \rho, T, n_i, \dot{n}_i, \dot{x}_{i,j}, T_j,$$

evaluated at P at time t , T denoting absolute temperature. OSEEN makes essentially the same assumption as to the variables which are relevant except that he does not use the principle and does allow values of n_i at particles different from P to enter into constitutive equations. ERICKSEN [1] deals with cases where the stress depends only on n_i and $\dot{x}_{i,j}$.

The relations discussed must satisfy an invariance requirement not considered by OSEEN [2], but now commonly used in continuum mechanics. It implies that these relations must be of the same form in all arbitrarily moving right-handed coordinate systems. This implies that (3.1) can be replaced by

$$(3.2) \quad \rho, T, n_i, \hat{n}_i, d_{ij}, T_j,$$

as can be established by reasoning similar to that used by ERICKSEN [1]. For remaining implications, we restrict our attention to the case where the functions considered are linear in the variables n_i , d_{ij} and T_j . Thus

$$(3.3) \quad \begin{aligned} t_{ij} &= A_{ij}^0 + A_{ijk}^1 \hat{n}_k + A_{ijkm}^2 d_{km} + A_{ijk}^3 T_k, \\ q_i &= B_i^0 + B_{ij}^1 \hat{n}_j + B_{ijk}^2 d_{jk} + B_{ij}^3 T_j, \\ g_i &= C_i^0 + C_{ij}^1 \hat{n}_j + C_{ijk}^2 d_{jk} + C_{ij}^3 T_j, \end{aligned}$$

where the A 's, B 's and C 's are functions of ϱ , T and n_i . In addition, we assume that these relations are invariant under reflections through all planes containing n_i as well as the rotations mentioned above. It is possible that this rules out some fluids to which the theory might otherwise apply. In general, such fluids would be optically active. In any event, these assumptions imply that the A 's, B 's and C 's are transversely isotropic tensors with respect to the direction n_i . Tensors of this type have been studied by SMITH & RIVLIN [9], who show that any such tensor is expressible as a linear combination of outer products formed from the tensors n_i and $\delta_{ij} - n_i n_j$ or, equivalently,

$$(3.4) \quad n_i, \quad \delta_{ij}.$$

For the situation discussed here, the scalar coefficients in these combinations can be shown to reduce to functions of

$$(3.5) \quad \varrho, \quad T, \quad n^2 = n_i n_i.$$

As a final simplification, we assume that

$$(3.6) \quad t_{ij} \rightarrow t_{ij}, \quad q_i \rightarrow q_i, \quad g_i \rightarrow -g_i$$

when we make the substitution

$$(3.7) \quad n_i \rightarrow -n_i \quad \text{and} \quad \hat{n}_i \rightarrow -\hat{n}_i,$$

i.e. that n_i and $-n_i$ are physically indistinguishable. Again, this may exclude some real fluids. From (3.3), (3.6), (3.7) and the results summarized in (3.4) and (3.5), we find by straightforward but somewhat laborious calculation that

$$(3.8) \quad \begin{aligned} t_{ij} = & (\alpha_0 + \alpha_1 d_{kk} + \alpha_2 d_{km} n_k n_m + \alpha_3 \hat{n}_k n_k) \delta_{ij} + \\ & + (\alpha_4 + \alpha_5 d_{kk} + \alpha_6 d_{km} n_k n_m + \alpha_7 \hat{n}_k n_k) n_i n_j + \\ & + \alpha_8 d_{ij} + \alpha_9 d_{ik} n_k n_j + \alpha_{10} d_{jk} n_k n_i + \alpha_{11} n_i \hat{n}_j + \alpha_{12} n_j \hat{n}_i, \end{aligned}$$

$$(3.9) \quad q_i = \beta_0 T_{,i} + \beta_1 n_k T_{,k} n_i,$$

$$(3.10) \quad g_i = (\gamma_0 + \gamma_1 d_{kk} + \gamma_2 d_{km} n_k n_m + \gamma_3 n_k \hat{n}_k) n_i + \gamma_4 d_{ij} n_j + \gamma_5 \hat{n}_i,$$

where the α 's, β 's and γ 's are functions of the variables listed in (3.5). Now it turns out that (2.13) must hold as an identity to keep the number of equations from exceeding the number of unknowns. This implies that

$$(3.11) \quad \gamma_4 \equiv \alpha_9 - \alpha_{10}, \quad \gamma_5 \equiv \alpha_{12} - \alpha_{11}.$$

4. Energetics

To complete the theory, we follow the time-honored procedure of assuming a local equation of state of the form

$$(4.1) \quad \varepsilon = \varepsilon(\varrho, n_i, S),$$

where S is entropy per unit mass, to be used in conjunction with the energy equation (2.17). For liquid crystals, it is customary to permit ε to depend on $n_{i,j}$ as well, as is discussed by FRANK [10]. We have the invariance requirement

that (4.1) retain its form under all rotations, which means that it reduces to the form

$$(4.2) \quad \varepsilon = \varepsilon(\varrho, n^2, S).$$

We assume as usual that

$$(4.3) \quad T = \frac{\partial \varepsilon}{\partial S} \geq 0,$$

and that this may be solved for S in terms of ϱ , n^2 and T . We may then introduce the free energy

$$(4.4) \quad \varphi = \varepsilon - TS = \varphi(\varrho, n^2, T),$$

such that

$$(4.5) \quad \left. \frac{\partial \varepsilon}{\partial \varrho} \right|_{S, n^2} = \left. \frac{\partial \varphi}{\partial \varrho} \right|_{T, n^2}, \quad \left. \frac{\partial \varepsilon}{\partial n^2} \right|_{S, \varrho} = \left. \frac{\partial \varphi}{\partial n^2} \right|_{T, \varrho}, \quad S = - \frac{\partial \varphi}{\partial T}.$$

From (2.15),

$$(4.6) \quad n_i \hat{n}_i = n_i (\dot{n}_i - w_{ij} n_j) = n_i \dot{n}_i,$$

since $w_{ij} = -w_{ji}$. From (2.9), (2.12), (4.2), (4.5) and (4.6), the rate of production of entropy is given by

$$\begin{aligned} \varrho T \dot{S} &= -\varrho T \frac{\partial \dot{\varphi}}{\partial T} = \varrho \frac{\partial \varepsilon}{\partial S} \dot{S} = \varrho \left(\dot{\varepsilon} - \frac{\partial \varepsilon}{\partial \varrho} \dot{\varrho} - \frac{\partial \varepsilon}{\partial n^2} 2n_i \dot{n}_i \right) \\ (4.7) \quad &= \varrho \left(\dot{\varepsilon} + \varrho \frac{\partial \varepsilon}{\partial \varrho} d_{kk} - 2 \frac{\partial \varepsilon}{\partial n^2} n_i \hat{n}_i \right) \\ &= \left(t_{ij} + \varrho^2 \frac{\partial \varepsilon}{\partial \varrho} \delta_{ij} \right) d_{ij} - \left(g_i + 2 \frac{\partial \varepsilon}{\partial n^2} n_i \right) \hat{n}_i - q_{i,i} \\ &= \left(t_{ij} + \varrho^2 \frac{\partial \varphi}{\partial \varrho} \delta_{ij} \right) d_{ij} - \left(g_i + 2 \frac{\partial \varphi}{\partial n^2} n_i \right) \hat{n}_i - q_{i,i}. \end{aligned}$$

The form of the Clausius inequality most often used in irreversible thermodynamics is

$$(4.8) \quad T[\varrho \dot{S} + (q_i/T)_{,i}] = \left(t_{ij} + \varrho^2 \frac{\partial \varphi}{\partial \varrho} \delta_{ij} \right) d_{ij} - \left(g_i + 2 \frac{\partial \varphi}{\partial n^2} n_i \right) \hat{n}_i - q_{i,i}/T \geq 0.$$

More explicit relations can of course be obtained using (3.8)–(3.10). This will lead to some restrictions on the coefficients occurring in (3.8)–(3.10) which we shall not investigate here.

5. Remarks

Equations (2.9), (2.10), (2.11) and (4.7) yield eight equations for the eight unknowns ϱ , T , $\dot{x}_i$ and n_i , it being understood that the body force and the relations (3.8), (3.9), (3.10) and (4.4) are specified. In general, it is consistent with these equations to take $n_i = 0$. The governing equations then reduce to those commonly used for isotropic fluids. Solutions with $n_i = 0$ are not always those which are most relevant. Under some conditions, solutions with $n_i \neq 0$ become unstable and there are stable solutions with $n_i \neq 0$. Our investigations of this situation are too incomplete to be included here.

Under the assumption that the "molecular inertia" $\varrho \ddot{n}_i$ is negligible, we can replace (2.11) by the equation $g_i = 0$, where g_i is given by (3.10). Ordinarily, this can be solved for $\hat{n}_i$, which yields equations of the form

$$(5.1) \quad \hat{n}_i = h_i(\varrho, T, n_k, d_{mp}).$$

Using this, we can eliminate $\hat{n}_i$ from the stress. ERICKSEN [1] introduces equations of this type, with ρ and T omitted, on the grounds that they seem to be the simplest type of equations which might be used to determine n_i . In this approximation, (2.3) yields $t_{ij} = t_{ji}$, as is assumed by ERICKSEN. Generally, inertia of suspended macromolecules or macroscopic particles is neglected in mechanistic theories of suspensions.

TRUESDELL [8] uses dimensional analysis to obtain restrictions on constitutive equations. This can yield additional restrictions in cases where one has *a priori* knowledge concerning the types of dimensional constants which can occur in the equations. Molecular theories might supply this. For the theory discussed here, one might appeal to molecular theories of suspensions of dumbbell molecules such as have been discussed by PRAGER [11] and GIESEKUS [12]. Except perhaps in certain limiting cases, there is not much similarity between the two types of theories, so we shall not pursue this. We impose no restrictions on the types of dimensional material constants which may occur. This means that the form which the basic equations take for some fixed choice of units is not restricted by requirements of dimensional invariance.

No unusual problems are encountered in modifying the present equations to obtain a simpler theory to describe isothermal flow of incompressible fluids. For this, we ignore (3.9) and (4.7) and replace the coefficient of δ_{ij} in (3.8) by an arbitrary scalar $-\bar{p}$. Terms proportional to $\bar{d}_{kk}$ will of course drop out because of the incompressibility condition. If we also adopt the approximation leading to (5.1), the governing equations reduce to the form

$$(5.2) \quad t_{ij} = t_{ji} = -\bar{p} \delta_{ij} + (\lambda_1 + \lambda_2 \bar{d}_{km} n_k n_m) n_i n_j + \lambda_3 \bar{d}_{ij} + \lambda_4 (\bar{d}_{ik} n_k n_j + \bar{d}_{jk} n_k n_i),$$

$$(5.3) \quad \hat{n}_i = (\mu_1 + \mu_2 \bar{d}_{km} n_k n_m) n_i + \mu_3 \bar{d}_{ij} n_j,$$

$$(5.4) \quad \dot{x}_{i,i} = \bar{d}_{i,i} = 0,$$

$$(5.5) \quad t_{ij,j} + f_i = \rho \ddot{x}_i.$$

Here the λ 's and μ 's are simple functions of the α 's and γ 's introduced earlier. For fixed ρ and T , they thus depend only on n^2 . The theory of ERICKSEN [1] treats n_i as a unit vector, which leads to further simplification.

To obtain properly invariant theories of anisotropic fluids, one must permit the stress to depend on something besides $\bar{d}_{ij}$, as is noted by NOLL [13, § A]. The simplest possibility seems to be to adjoin a vector, as is done here. In somewhat similar theories of solids, triads of vectors have been introduced by several writers. Information concerning this is given by ERICKSEN & TRUESDELL [14]. Introducing more vectors or higher order tensors would enable one to incorporate in theories of fluids more detailed descriptions of complex molecular structures. For polyatomic gases, it is conceivable that theories of this type might be useful in describing behavior observed in ultrasonic experiments. As can be seen from the work of TRUESDELL [15], the Navier-Stokes equations seem to be rather successful in predicting the behavior of monatomic gases.

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References

- [1] ERICKSEN, J. L.: Transversely isotropic fluids (to appear in Rheol. Acta).
- [2] OSEEN, C. W.: The theory of liquid crystals. Trans. Faraday Soc. **29**, 883—899 (1933).
- [3] GREEN, A. E., & R. S. RIVLIN: The mechanics of non-linear materials with memory. Part I. Arch. Rational Mech. Anal. **1**, 1—34 (1957).
- [4] NOLL, W.: A mathematical theory of the mechanical behavior of continuous media. Arch. Rational Mech. Anal. **2**, 198—226 (1958).
- [5] GREEN, A. E., R. S. RIVLIN & J. M. SPENCER: The mechanics of nonlinear materials with memory, Part II. Arch. Rational Mech. Anal. **3**, 82—90 (1959).
- [6] TOUPIN, R. A.: The elastic dielectric. J. Rational Mech. Anal. **5**, 849—916 (1956).
- [7] GRAD, H.: Statistical Mechanics, thermodynamics, and fluid dynamics of systems with an arbitrary number of integrals. Comm. Pure Appl. Math. **5**, 455—494 (1952).
- [8] TRUESDELL, C.: A new definition of a fluid II. The Maxwellian fluid. J. Math. Pures Appl. (9) **30**, 115—158 (1951).
- [9] SMITH, G. F., & R. S. RIVLIN: The anisotropic tensors. Quart. Appl. Math. **15**, 308—314 (1957).
- [10] FRANK, F. C.: On the theory of liquid crystals. Discussions Faraday Soc. **25**, 19—28 (1958).
- [11] PRAGER, S.: Stress-strain relations in a suspension of dumbbells. Trans. Soc. Rheol. **1**, 53—62 (1957).
- [12] GIESEKUS, H.: Die rheologische Zustandsgleichung. Rheol. Acta **1**, 2—20 (1958).
- [13] NOLL, W.: On the continuity of the solid and fluid states. J. Rational Mech. Anal. **4**, 3—81 (1955).
- [14] ERICKSEN, J. L., & C. TRUESDELL: Exact theory and stress and strain in rods and shells. Arch. Rational Mech. Anal. **1**, 295—323 (1958).
- [15] TRUESDELL, C.: Precise theory of the absorption and dispersion of forced plane infinitesimal waves according to the Navier-Stokes equations. J. Rational Mech. Anal. **2**, 643—741 (1953).

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The Distribution of Thermal Stress in an Infinite Elastic Solid Containing a Penny-Shaped Crack

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1. Introduction

In recent years there has been great interest in the distribution of stress in the neighbourhood of Griffith or penny-shaped cracks in the interior of an elastic body, mainly because of its importance in the theory of fracture. An excellent review of these calculations is given by IRWIN (1958). Although, at the same time, there has been a revival of interest in the calculation of thermal stresses (GOODIER 1958), the problem of calculating the thermal stresses in the neighbourhood of a crack in an elastic body does not appear to have been considered. In this paper consideration is given to the calculation of the distribution of stress in the neighbourhood of a penny-shaped crack of the type considered previously by SNEDDON (1946). Similar calculations have been begun on the distribution of stress in the neighbourhood of a Griffith crack, but these will be reported in a separate communication. It is assumed that the stress is set up by the application of heat to the surfaces of the crack.

The use of the classical equations of elastic equilibrium restricts the application of the analysis to materials which obey Hooke's law fairly closely, and to thermal disturbances which lead only to infinitesimal strains. These disturbances are also required to be constant with respect to the time. By making the further assumption that the thermal conditions on the upper surface of the crack are identical with those on the lower surface, we are able to reduce the problem to one concerning the distribution of thermal stress in a semi-infinite elastic solid and therefore to make use of the methods developed in the earlier paper (SNEDDON 1946). The solution so obtained will give an accurate picture of the distribution of thermal stress in an elastic body which is so large that its linear dimensions are very much greater than the radius of the crack. For simplicity we assume that the body is isotropic (though the method can easily be extended to the anisotropic case) and homogeneous.

In §§ 2 and 3 the solution of the statical equations of thermoelasticity is derived for problems concerning a semi-infinite elastic solid in which there is symmetry about an axis normal to the boundary plane of the solid. In § 4 this solution is used to determine the distribution of temperature and stress in an infinite elastic solid containing a penny-shaped crack of radius a , across whose surfaces there is a prescribed flux of heat $Q(r/a)$ which is a function of the radial

distance r from the centre of the crack. Two special cases are considered in detail in § 5. In the first case $Q(r/a)$ is a constant. In the second case it is assumed that the function $Q(r/a)$ is such that it possesses a Fourier-Bessel expansion of the type $\sum_m Q_m J_0(j_m r/a)$, where $j_1, j_2, \dots$ are the positive zeros of the Bessel function $J_0(x)$. In this latter case we obtain the striking result that the normal component of stress is of the form $Kf(r/a, z/a)$, where r and z are cylindrical coordinates and the function f is independent of the Fourier-Bessel coefficients $Q_1, Q_2, \dots$ of the flux function; the constant K does however involve these coefficients. It can in fact be shown that for flux functions possessing such an expansion that K is proportional to $\int_0^1 s Q(s) ds$. A table of values of the function f is given and its variation with r and z illustrated graphically in Fig. 1. This unusual behaviour of the normal component of stress σ_z is reminiscent of the degeneracy first noted by STERNBERG & McDOWELL (1957) in their analysis of the thermal stresses in a semi-infinite solid whose surface is free from stress; they found that σ_z is identically zero at every point of the solid.

Finally we consider the distribution of stress produced by prescribing the temperature at the surface of the crack. By calculating the equivalent flux function necessary to produce this prescribed temperature, we reduce the problem to that considered in § 4. To illustrate the method, we derive the solution for the case in which the surface of the crack is maintained at a constant temperature.

2. Solution of the Equations of Thermoelasticity

We shall consider the temperature and displacement fields in a perfectly elastic solid which is conducting heat. With regard to both its mechanical and its thermal properties the solid will be assumed to be isotropic and homogeneous. In the problems which we shall consider in this paper it will be assumed that there is symmetry about an axis, which, for convenience, we shall take to be the z -axis. The position of a typical point of the solid may then conveniently be expressed by a triplet of cylindrical coordinates (r, φ, z) . For symmetrical deformations of the solid the displacement vector $\mathbf{u}$ of such a point may be taken to have components $(u, 0, w)$ in this coordinate system, and the only non-vanishing components of the stress tensor will be $\sigma_r, \sigma_\varphi, \sigma_z$ and τ_{rz} .

The equations of equilibrium of an elastic solid conducting heat may be written in this coordinate system in the forms (SNEDDON & BERRY 1958, p. 125)

$$2(1-\nu) \left(\frac{\partial^2 u}{\partial r^2} + \frac{1}{r} \frac{\partial u}{\partial r} - \frac{u}{r^2} \right) + (1-2\nu) \frac{\partial^2 u}{\partial z^2} + \frac{\partial^2 w}{\partial r \partial z} = 2(1+\nu) \alpha \frac{\partial \vartheta}{\partial r}, \quad (1)$$

$$(1-2\nu) \left(\frac{\partial^2 w}{\partial r^2} + \frac{1}{r} \frac{\partial w}{\partial r} \right) + 2(1-\nu) \frac{\partial^2 w}{\partial z^2} + \frac{\partial}{\partial z} \left(\frac{\partial u}{\partial r} + \frac{u}{r} \right) = 2(1+\nu) \alpha \frac{\partial \vartheta}{\partial z}, \quad (2)$$

where ϑ is the deviation of the absolute temperature from the temperature of the solid in a state of zero stress and strain, α is the coefficient of linear expansion of the solid and ν is its Poisson ratio. The temperature field in the solid is determined by Laplace's equation

$$\frac{\partial^2 \vartheta}{\partial r^2} + \frac{1}{r} \frac{\partial \vartheta}{\partial r} + \frac{\partial^2 \vartheta}{\partial z^2} = 0, \quad (3)$$

in the steady state, and in the absence of thermal sources.

We can readily reduce this set of partial differential equations to a set of ordinary differential equations by introducing Hankel transforms of the independent variables u , w and ϑ . We define the first order Hankel transform of the radial component of the displacement vector by the equation

$$\bar{u}(\xi, z) = \int_0^\infty r u(r, z) J_1(\xi r) dr, \quad (4)$$

and the zero order Hankel transforms of the z -component of the displacement and the temperature by the equations

$$\bar{w}(\xi, z) = \int_0^\infty r w(r, z) J_0(\xi r) dr, \quad (5)$$

$$\bar{\vartheta}(\xi, z) = \int_0^\infty r \vartheta(r, z) J_0(\xi r) dr. \quad (6)$$

If we multiply both sides of equation (1) by $r J_1(\xi r)$, integrate with respect to r from 0 to ∞ , and make use of well-known properties of Hankel transforms (SNEDDON 1951, p. 60), we find that this equation is equivalent to the second order ordinary differential equation

$$[(1 - 2\nu) D^2 - (2 - 2\nu) \xi^2] \bar{u} - \xi D \bar{w} = -2(1 + \nu) \alpha \xi \bar{\vartheta}, \quad (7)$$

in which D denotes the operator d/dz . Similarly, if we multiply both sides of equations (2) and (3) by $r J_0(\xi r)$ and integrate over all positive r , we find that these partial differential equations are equivalent to the pair of ordinary differential equations

$$[2(1 - \nu) D^2 - (1 - 2\nu) \xi^2] \bar{w} + \xi D \bar{u} = 2(1 + \nu) \alpha D \bar{\vartheta}, \quad (8)$$

$$(D^2 - \xi^2) \bar{\vartheta} = 0. \quad (9)$$

If we eliminate $\bar{u}$ and $\bar{w}$ from equations (8) and (7) in turn and make use of equation (9), we find that

$$(D^2 - \xi^2)^2 \bar{u} = 0, \quad (D^2 - \xi^2)^2 \bar{w} = 0. \quad (10)$$

The boundary values appropriate to any given problem can similarly be transformed to boundary values on the Hankel transforms $\bar{u}$, $\bar{w}$ and $\bar{\vartheta}$, and the equations (8) - (10) solved to give the expressions for these Hankel transforms. From these expressions we obtain the required solution by making use of the Hankel inversion theorem (SNEDDON 1951, p. 47) to obtain the equations

$$u(r, z) = \int_0^\infty \xi \bar{u}(\xi, z) J_1(\xi r) d\xi, \quad (11a)$$

$$w(r, z) = \int_0^\infty \xi \bar{w}(\xi, z) J_0(\xi r) d\xi, \quad (11b)$$

$$\vartheta(r, z) = \int_0^\infty \xi \bar{\vartheta}(\xi, z) J_0(\xi r) d\xi. \quad (11c)$$

3. Solution for a Semi-Infinite Solid

In the case of a semi-infinite solid $z \geq 0$, assumed free from disturbance at infinity, we are interested in solutions of the equations (1)–(3) which tend to zero as $z \rightarrow \infty$, so that the appropriate solution of equation (9) is

$$\bar{\vartheta}(\xi, z) = A(\xi) e^{-\xi z}, \quad (12)$$

where $A(\xi)$, which is a function only of the parameter ξ , is determined from the boundary conditions of the temperature field. If we substitute this expression for $\bar{\vartheta}$ into equations (7) and (8), we obtain the pair of simultaneous ordinary linear differential equations

$$\begin{aligned} [(1-2\nu)D^2 - 2(1-\nu)\xi^2]\bar{u} - \xi D\bar{w} &= -2(1+\nu)\alpha\xi A(\xi)e^{-\xi z}, \\ [2(1-\nu)D^2 - (1-2\nu)\xi^2]\bar{w} + \xi D\bar{u} &= -2(1+\nu)\alpha\xi A(\xi)e^{-\xi z}, \end{aligned}$$

which are easily seen to have solutions

$$\bar{u} = (A_1 + \xi z B_1) e^{-\xi z}, \quad \bar{w} = (A_2 + \xi z B_2) e^{-\xi z}, \quad (13)$$

where

$$B_1 = B_2, \quad A_1 - A_2 + (3-4\nu)B_2 = 2(1+\nu)\alpha\xi^{-1}A(\xi). \quad (14)$$

To find the further necessary relations among A_1 , B_1 , A_2 and B_2 , we shall have to make use of the boundary conditions. The shearing stress τ_{rz} is determined in terms of the displacement vector by the equation

$$\tau_{rz} = \mu \left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial r} \right).$$

If we multiply both sides of this equation by $r J_1(\xi r)$ and integrate with respect to r from 0 to ∞ , we find that

$$\begin{aligned} \int_0^\infty r \tau_{rz} J_1(\xi r) dr &= \mu (D\bar{u} - \xi \bar{w}) \\ &= -\mu \xi [A_1 - B_1 + A_2 + (B_1 + B_2)\xi z] e^{-\xi z}. \end{aligned}$$

The boundary condition

$$\tau_{rz} = 0 \quad \text{on} \quad z = 0 \quad (15)$$

therefore implies the relation

$$A_1 - B_1 + A_2 = 0. \quad (16)$$

From the three equations (14) and (16) we find that

$$B_1 = B_2 = \frac{1}{2(1-\nu)} A_2 + \frac{1+\nu}{2(1-\nu)} \alpha \xi^{-1} A(\xi), \quad (17)$$

$$A_1 = -\frac{1-2\nu}{2(1-\nu)} A_2 + \frac{1+\nu}{2(1-\nu)} \alpha \xi^{-1} A(\xi). \quad (18)$$

Now from the Duhamel-Neumann stress-strain relation (SNEDDON & BERRY 1958, p. 123), we know that

$$\sigma_z = 2\mu \left\{ \frac{1-\nu}{1-2\nu} \frac{\partial w}{\partial z} + \frac{\nu}{1-2\nu} \left(\frac{\partial u}{\partial r} + \frac{u}{r} \right) - \frac{1+\nu}{1-2\nu} \alpha \vartheta \right\},$$

from which it follows that the Hankel transform,

$$\bar{\sigma}_z = \int_0^\infty r J_0(\xi r) \sigma_z dr, \quad (19)$$

of the normal stress σ_z is given in terms of $\bar{u}$, $\bar{w}$ and $\bar{\vartheta}$ by the equation

$$\bar{\sigma}_z = \frac{2\mu}{1-2\nu} [(1-\nu) D \bar{w} + \nu \xi \bar{u} - (1+\nu) \alpha \bar{\vartheta}]. \quad (20)$$

Substituting from equations (12) and (13) into this relation, we find that

$$\bar{\sigma}_z = -\frac{2\mu}{1-2\nu} \xi e^{-\xi z} \{ (1-\nu) (A_2 - B_2) - \nu A_1 + (1+\nu) \alpha \xi^{-1} A + [(1-\nu) B_2 - \nu B_1] \xi z \},$$

and, making use of equations (17) and (18), we see that this is equivalent to the relation

$$\bar{\sigma}_z = -\frac{\mu}{1-\nu} [A_2 + (1+\nu) \alpha \xi^{-1} A] (1 + \xi z) \xi e^{-\xi z}. \quad (21)$$

Inverting the expressions (12), (13) and (21) by Hankel's inversion theorem, we obtain the expressions

$$u = \frac{1}{2(1-\nu)} \int_0^\infty \{ [\xi z - (1-2\nu)] A_2(\xi) + (1+\nu) (1 + \xi z) \alpha \xi^{-1} A(\xi) \} e^{-\xi z} J_1(\xi r) \xi d\xi,$$

$$w = \int_0^\infty \left\{ A_2(\xi) + \frac{\xi z}{2(1-\nu)} [A_2(\xi) + (1+\nu) \alpha \xi^{-1} A(\xi)] \right\} e^{-\xi z} J_0(\xi r) \xi d\xi,$$

$$\vartheta = \int_0^\infty \xi A(\xi) e^{-\xi z} J_0(\xi r) d\xi$$

for the components of the displacement vector and the temperature. We note that, for this solution, when $z=0$,

$$\tau_{rz} = 0, \quad \sigma_z = -\frac{\mu}{1-\nu} \int_0^\infty \xi^2 [A_2 + (1+\nu) \alpha \xi^{-1} A(\xi)] J_0(r\xi) d\xi.$$

It is convenient to write the solution in a dimensionless form. Putting $(1+\nu) \alpha A(\xi) = a \varphi(\xi a)$, $A_2(\xi) = (a/\xi) \psi(a\xi)$, where a is some typical length, and changing to a new variable of integration $\eta = \xi a$, we find that the solution we have obtained may be written in the form

$$u = \frac{1}{2(1-\nu)} \int_0^\infty \{ [\eta \zeta - (1-2\nu)] \psi(\eta) + (1+\eta \zeta) \varphi(\eta) \} J_1(\varrho \eta) e^{-\zeta \eta} d\eta, \quad (22)$$

$$w = \int_0^\infty \left\{ \psi(\eta) + \frac{\eta \zeta}{2(1-\nu)} [\psi(\eta) + \varphi(\eta)] \right\} J_0(\varrho \eta) e^{-\zeta \eta} d\eta, \quad (23)$$

$$\vartheta = \frac{1}{(1+\nu) \alpha a} \int_0^\infty \eta \varphi(\eta) J_0(\varrho \eta) e^{-\zeta \eta} d\eta, \quad (24)$$

where

$$\varrho = r/a, \quad \zeta = z/a. \quad (25)$$

This solution has the property that, when $z=0$,

$$w = \int_0^{\infty} \psi(\eta) J_0(\varrho\eta) d\eta, \quad (26)$$

$$\vartheta = \frac{1}{(1+\nu)\alpha a} \int_0^{\infty} \eta \varphi(\eta) J_0(\varrho\eta) d\eta, \quad (27)$$

$$\frac{\partial \vartheta}{\partial z} = - \frac{1}{(1+\nu)\alpha a^2} \int_0^{\infty} \eta^2 \varphi(\eta) J_0(\varrho\eta) d\eta, \quad (28)$$

$$\tau_{rz} = 0, \quad (29)$$

$$\sigma_z = - \frac{\mu}{(1-\nu)a} \int_0^{\infty} \eta [\psi(\eta) + \varphi(\eta)] J_0(\varrho\eta) d\eta. \quad (30)$$

Combining equations (30) and (27) we see that, for this solution,

$$\sigma_z = - \frac{\mu}{(1-\nu)a} \int_0^{\infty} \eta \psi(\eta) J_0(\varrho\eta) d\eta - \frac{1+\nu}{1-\nu} \alpha \mu \vartheta, \quad \text{on } z=0. \quad (31)$$

In general, we have

$$\sigma_z = - \frac{\mu}{(1-\nu)a} \int_0^{\infty} [\psi(\eta) + \varphi(\eta)] \eta (1+\eta\zeta) e^{-\zeta\eta} J_0(\varrho\eta) d\eta. \quad (32)$$

The determination of the unknown functions $\psi(\eta)$, $\varphi(\eta)$ can be achieved only if we further specify the mechanical and thermal conditions on the surface of the solid.

For instance, if the plane boundary $z=0$ is free from applied stress, then $\tau_{rz}=\sigma_z=0$ on that plane, and we see that we must take $\psi(\eta)=-\varphi(\eta)$, so that the appropriate solution is given by the equations

$$\begin{aligned} u &= \int_0^{\infty} \varphi(\eta) J_1(\varrho\eta) e^{-\zeta\eta} d\eta, \\ w &= - \int_0^{\infty} \varphi(\eta) J_0(\varrho\eta) e^{-\zeta\eta} d\eta, \\ \vartheta &= \frac{1}{(1+\nu)\alpha a} \int_0^{\infty} \eta \varphi(\eta) J_0(\varrho\eta) e^{-\zeta\eta} d\eta. \end{aligned}$$

The function $\varphi(\eta)$ is determined by the thermal conditions. For example, if, on $z=0$, it is prescribed that $\vartheta=\vartheta_0 f(r/a)$, we find that $\varphi(\eta)=(1+\nu)\alpha\vartheta_0 a \bar{f}(\eta)$ where $\bar{f}(\eta)$ is the zero-order Hankel transform of the function $f(\varrho)$. Hence we obtain the solution

$$\begin{aligned} u &= (1+\nu)\alpha\vartheta_0 a \int_0^{\infty} \bar{f}(\eta) J_1(\varrho\eta) e^{-\zeta\eta} d\eta, \\ w &= - (1+\nu)\alpha\vartheta_0 a \int_0^{\infty} \bar{f}(\eta) J_0(\varrho\eta) e^{-\zeta\eta} d\eta. \end{aligned}$$

This is the axially symmetrical solution of the problem considered by STERNBERG & McDOWELL (1957) and by SNEDDON & LOCKETT (1960), and, as proved by these authors, we find that for all values of r and z , $\sigma_z=\tau_{rz}=0$.

4. Prescribed Flux of Heat across the Surfaces of a Penny-Shaped Crack

In this section we shall suppose that a penny-shaped crack, situated in an infinite elastic solid, is opened out by the application of heat to its flat surfaces. We shall assume that the flux of heat across the upper surface of the crack is exactly the same as that across the lower surface, and that its dependence on r is known. We suppose that the crack is created in the interior of an infinite medium and that it is penny-shaped, occupying the circular area $r \leq a$ in the plane $z = 0$. If we assume that the crack is deformed by the application of an axially symmetrical pressure (in addition to the heat flux), then we see that, when $z = 0$, $\sigma_z = -p(r/a)$, a prescribed function of r/a defined for $r \leq a$. As in a previous paper (SNEDDON 1946), we see that we may assume that the distribution of stress and temperature in the neighbourhood of a crack in an infinite elastic solid is identical with that produced in a semi-infinite elastic medium $z \geq 0$, when its boundary surface $z = 0$ is subjected to the boundary conditions

$$\frac{\partial \vartheta}{\partial z} = Q(r/a), \quad 0 \leq r < a, \quad (33)$$

$$\frac{\partial \vartheta}{\partial z} = 0, \quad r > a, \quad (34)$$

$$w = 0, \quad r > a, \quad (35)$$

$$\sigma_z = -p(r/a), \quad 0 \leq r < a. \quad (36)$$

From equations (33) and (34) and equation (28) above, we find, on applying the Hankel inversion theorem, that the appropriate form of the function $\varphi(\eta)$ is given by the equation

$$\eta \varphi(\eta) = -(1+\nu) \alpha a^2 \int_0^1 \varrho Q(\varrho) J_0(\varrho \eta) d\varrho$$

which we may write in the form

$$\varphi(\eta) = -(1+\nu) \alpha a^2 \eta^{-1} q(\eta), \quad (37)$$

where

$$q(\eta) = \int_0^1 \varrho Q(\varrho) J_0(\varrho \eta) d\varrho. \quad (38)$$

It follows from equation (24) that

$$\vartheta = -a \int_0^\infty q(\eta) J_0(\varrho \eta) e^{-\xi \eta} d\eta. \quad (39)$$

Substituting from equations (36) and (39) into equation (31), we see that, when $z = 0$, $0 < \varrho < 1$,

$$-\frac{\mu a}{1-\nu} \int_0^\infty \eta \psi(\eta) J_0(\varrho \eta) d\eta + \frac{(1+\nu) \alpha a^3 \mu}{1-\nu} \int_0^\infty q(\eta) J_0(\varrho \eta) d\eta = -p(\varrho).$$

Hence, if we write

$$f(\varrho) = \frac{1-\nu}{\mu a} p(\varrho) + (1+\nu) \alpha a^2 \int_0^\infty q(\eta) J_0(\varrho \eta) d\eta$$

and make use of condition (35) with equation (26), we see that $\psi(\eta)$ must satisfy the dual integral equations

$$\begin{aligned}\int_0^{\infty} \eta \psi(\eta) J_0(\varrho \eta) d\eta &= f(\varrho), & 0 < \varrho < 1, \\ \int_0^{\infty} \psi(\eta) J_0(\varrho \eta) d\eta &= 0, & \varrho > 1.\end{aligned}$$

The solution of these equations is known to be (SNEDDON 1951, p. 65)

$$\psi(\eta) = \frac{2}{\pi} \int_0^1 \sin(\eta s) ds \int_0^s \frac{x f(x) dx}{V(s^2 - x^2)}. \quad (40)$$

Now, interchanging the orders of integration and making use of standard properties of Bessel functions, we find that

$$\begin{aligned}\int_0^s \frac{x dx}{V(s^2 - x^2)} \int_0^{\infty} q(\eta) J_0(x\eta) d\eta &= \int_0^{\infty} \frac{\sin s \eta}{\eta} d\eta \int_0^1 \varrho Q(\varrho) J_0(\varrho \eta) d\varrho, \\ &= \int_0^1 \varrho Q(\varrho) d\varrho \int_0^{\infty} \frac{\sin s \eta}{\eta} J_0(\varrho \eta) d\eta, \\ &= \int_s^1 \varrho Q(\varrho) \sin^{-1}\left(\frac{s}{\varrho}\right) d\varrho + \frac{1}{2} \pi \int_0^s \varrho Q(\varrho) d\varrho.\end{aligned}$$

Applying the rule for integration by parts to this last integral, and introducing the function

$$\mathcal{Q}(\varrho) = \int_{\varrho}^1 \varrho Q(\varrho) d\varrho, \quad (41)$$

we find that this integral is equal to

$$\frac{1}{2} \pi \mathcal{Q}(0) - s \int_s^1 \frac{Q(\varrho) d\varrho}{\varrho V(\varrho^2 - s^2)}$$

and, after a little manipulation, that

$$\begin{aligned}\psi(\eta) &= \frac{2(1-\nu)}{\pi \mu a} \int_0^1 \sin(\eta s) ds \int_0^s \frac{x \phi(x) dx}{V(s^2 - x^2)} + \\ &+ (1+\nu) \alpha a^2 \left\{ \mathcal{Q}(0) \frac{1 - \cos \eta}{\eta} - \int_0^1 \mathcal{Q}(\varrho) J_1(\varrho \eta) d\varrho \right\}.\end{aligned} \quad (42)$$

We obtain the solution of the problem by substituting from equations (42) and (37) into equations (22), (23) and (24).

5. Special Problems

To illustrate the use of the results of the last section, we shall now consider the distribution of stress produced in the elastic solid when the flux function takes two special forms. In the first case we shall assume that the function

$Q(r/a)$ is a constant, Q_0 say, and in the second that it can be represented by a Fourier-Bessel series. In both cases we shall assume that there is no external pressure applied to the surface of the crack, so that $p(r/a) = 0$. This is not at all a severe limitation, since the solution for the case of a non-zero function $p(r/a)$ is known for a solid with $\alpha = 0$ (SNEDDON 1946) and the general solution may be obtained by superposition.

Case (i): $Q = Q_0$, a constant.

If we put $Q(\varrho) = Q_0$ in equation (41) we find that

$$\mathcal{Q}(\varrho) = \frac{1}{2} Q_0 (1 - \varrho^2). \quad (43)$$

From equation (42) we find that

$$\psi(\eta) = (1 + \nu) \alpha a^2 Q_0 \left\{ \frac{J_1(\eta)}{\eta^2} - \frac{\cos \eta}{2\eta} \right\} \quad (44)$$

and from equations (37) and (38) that

$$\varphi(\eta) = - (1 + \nu) \alpha a^2 Q_0 \frac{J_1(\eta)}{\eta^2}. \quad (45)$$

If we substitute from equations (44) and (45) into equation (22), we obtain the expression

$$u = \frac{(1 + \nu) \alpha a^2 Q_0}{2(1 - \nu)} \int_0^\infty \left\{ [\eta \zeta - (1 - 2\nu)] [\eta^{-2} J_1(\eta) - \frac{1}{2} \eta^{-1} \cos \eta] - (1 + \eta \zeta) \eta^{-2} J_1(\eta) \right\} J_1(\varrho \eta) e^{-\zeta \eta} d\eta \quad (46)$$

for the radial component of the displacement vector. Similarly, from equation (23), we obtain the expression

$$w = \frac{1}{2} (1 + \nu) \alpha a^2 Q_0 \int_0^\infty \left\{ \frac{2J_1(\eta)}{\eta^2} - \frac{\cos \eta}{\eta} - \frac{\zeta}{4(1 - \nu)} \cos \eta \right\} J_0(\varrho \eta) e^{-\zeta \eta} d\eta \quad (47)$$

for the other non-vanishing component of the displacement vector.

It follows from equation (47) that the normal displacement of the surface of the crack is obtained by putting $\zeta = 0$, $0 < \varrho < 1$ in this integral to obtain the expression

$$w_0 = \frac{1}{2} (1 + \nu) \alpha a^2 Q_0 \int_0^\infty \frac{1 - \cos \eta}{\eta} J_0(\varrho \eta) d\eta + \frac{1}{2} (1 + \nu) \alpha a^2 Q_0 \int_0^\infty \left(\frac{2J_1(\eta)}{\eta} - 1 \right) \frac{J_0(\varrho \eta)}{\eta} d\eta.$$

Now

$$\int_0^\infty \frac{1 - \cos \eta}{\eta} J_0(\varrho \eta) d\eta = \begin{cases} \cosh^{-1}(1/\varrho), & 0 < \varrho \leq 1, \\ 0, & \varrho > 1, \end{cases}$$

and

$$\alpha \int_0^\infty \frac{1 - J_0(\alpha \eta)}{\eta} J_0(\varrho \eta) d\eta = \begin{cases} \alpha \log_e(\alpha/\varrho), & \varrho < \alpha, \\ 0, & \varrho > \alpha \end{cases}$$

so that, integrating both sides of the last result with respect to α from 0 to 1, we obtain the result

$$\int_0^\infty \left\{ \frac{1}{2\eta} - \frac{J_1(\eta)}{\eta^2} \right\} J_0(\varrho\eta) d\eta = -\frac{1}{2} \log_e \varrho - \frac{1}{4} (1 - \varrho^2).$$

Using these results and the fact that

$$\cosh^{-1} \left(\frac{1}{\varrho} \right) + \log_e \varrho = \log_e [1 + \sqrt{1 - \varrho^2}], \quad 0 < \varrho \leq 1,$$

we find that, when $0 < \varrho < 1$,

$$\int_0^\infty \frac{1 - \cos \eta}{\eta} J_0(\varrho\eta) d\eta + \int_0^\infty \left(\frac{2J_1(\eta)}{\eta^2} - \frac{1}{\eta} \right) J_0(\varrho\eta) d\eta = \log_e [1 + \sqrt{1 - \varrho^2}] + \frac{1}{2} (1 - \varrho^2)$$

so that, finally, we have

$$w_0 = \frac{1}{4} (1 + \nu) \alpha a^2 Q_0 \{ 2 \log_e [1 + \sqrt{1 - \varrho^2}] + (1 - \varrho^2) \}, \quad 0 < \varrho < 1. \quad (48)$$

It will be observed that, when $\varrho = 1$, $w_0 = 0$, and that, when $\varrho = 0$,

$$w_0 = \frac{1}{4} (1 + \nu) \alpha a^2 Q_0 (2 \log_e 2 + 1). \quad (49)$$

Once we have found the integral expressions for the components of the displacement vector, we can readily find the corresponding expressions for the components of the stress tensor by differentiating under the integral sign. We shall illustrate the procedure involved by evaluating the normal component of stress σ_z . If we substitute from equations (44) and (45) into equation (32), we find that

$$\sigma_z = -\frac{\mu(1+\nu)}{2(1-\nu)} \alpha a Q_0 f(\varrho, \zeta), \quad (50)$$

where the function $f(\varrho, \zeta)$ is defined by the integral

$$f(\varrho, \zeta) = \int_0^\infty (1 + \eta \zeta) e^{-\zeta \eta} \cos \eta J_0(\varrho \eta) d\eta. \quad (51)$$

This integral can be easily evaluated to give

$$f(\varrho, \zeta) = \frac{1}{R} \cos \frac{1}{2} \varphi + \frac{\zeta}{R^3} \left[\zeta \cos \frac{3}{2} \varphi + \sin \frac{3}{2} \varphi \right], \quad (52)$$

with

$$R^4 = (\varrho^2 + \zeta^2 - 1)^2 + 4\zeta^2, \quad \tan \varphi = \frac{2\zeta}{\varrho^2 + \zeta^2 - 1}. \quad (53)$$

It should be observed that

$$f(0, \zeta) = \frac{2\zeta^3}{(1 + \zeta^2)^2}$$

and that

$$f(\varrho, 0) = \begin{cases} 0, & 0 < \varrho < 1, \\ (\varrho^2 - 1)^{-\frac{1}{2}}, & \varrho > 1. \end{cases}$$

The variation of $f(\varrho, \zeta)$ with ϱ and ζ for a set of values of ϱ and ζ is shown in Table 1. This variation is also shown graphically in Fig. 1. From the values of this function we can easily calculate the normal component of stress σ_z , by means of equation (50).

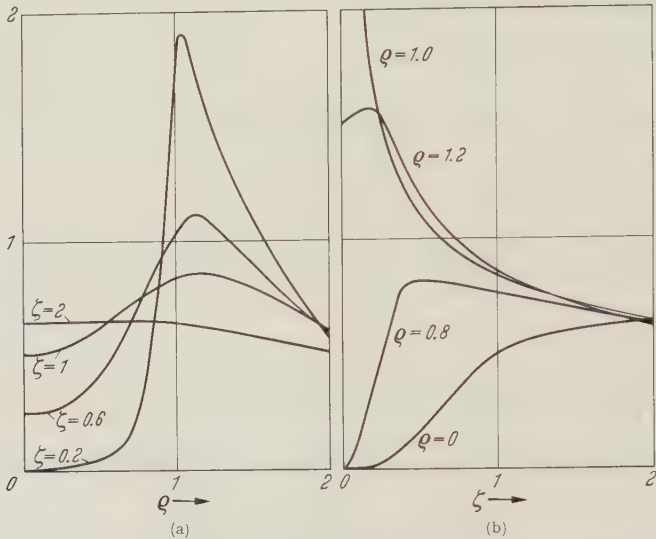


Fig. 1. The variation of $f(\varrho, \zeta)$ with ϱ and ζ . (a) shows the form of $f(\varrho, \zeta)$ as ϱ varies, ζ being kept fixed, and (b) shows the form of $f(\varrho, \zeta)$ as ζ varies, ϱ being kept fixed

Table 1. Values of the function $f(\varrho, \zeta)$

$\varrho \backslash \zeta$	0	0.2	0.4	0.6	0.8	1	2	3	4	5	10
0	0	0.0148	0.0951	0.2336	0.3807	0.5000	0.6400	0.5400	0.4429	0.3698	0.1961
0.2	0	0.0174	0.1097	0.2578	0.4057	0.5199	0.6410	0.5391	0.4422	0.3694	0.1960
0.4	0	0.0305	0.1666	0.3407	0.4833	0.5782	0.6433	0.5365	0.4404	0.3681	0.1958
0.6	0	0.0832	0.3214	0.5132	0.6149	0.6670	0.6453	0.5320	0.4369	0.3659	0.1954
0.8	0	0.3763	0.7915	0.8113	0.7799	0.7656	0.6464	0.5255	0.4322	0.3629	0.1949
1	∞	1.7075	1.2185	1.0314	0.9161	0.8399	0.6396	0.5169	0.4263	0.3591	0.1943
1.2	1.5076	1.5800	1.3166	1.0990	0.9558	0.8614	0.6281	0.5062	0.4193	0.3546	0.1935
2	0.5774	0.5844	0.5997	0.6120	0.6152	0.6092	0.5175	0.4454	0.3812	0.3301	0.1890
3	0.3536	0.3547	0.3581	0.3625	0.3679	0.3727	0.3769	0.3539	0.3222	0.2906	0.1810
4	0.2582	0.2586	0.2598	0.2616	0.2641	0.2666	0.2774	0.2766	0.2656	0.2495	0.1740
5	0.2041	0.2043	0.2049	0.2058	0.2070	0.2084	0.2179	0.2226	0.2190	0.2124	0.1596
10	0.1005	0.1005	0.1006	0.1007	0.1009	0.1010	0.1024	0.1043	0.1063	0.1079	0.1061

Case (ii). $Q(r/a)$ represented by a Fourier-Bessel Series.

If the flux function $Q(\varrho)$ is represented by the Fourier-Bessel expansion

$$Q(\varrho) = \sum_{m=1}^{\infty} Q_m J_0(j_m \varrho), \tag{54}$$

where $j_1, j_2, \dots$ are the positive zeros of the Bessel function $J_0(z)$, then, by equation (41),

$$\mathcal{Q}(\varrho) = \sum_{m=1}^{\infty} \frac{Q_m}{j_m} [J_1(j_m) - \varrho J_1(j_m \varrho)], \tag{55}$$

with

$$\mathcal{Q}(0) = \sum_{m=1}^{\infty} \frac{Q_m J_1(j_m)}{j_m}. \quad (56)$$

From equations (37), (38) and (42), we find that

$$\varphi(\eta) = -(1+\nu) \alpha a^2 J_0(\eta) \sum_{m=1}^{\infty} \frac{j_m Q_m J_1(j_m)}{\eta (j_m^2 - \eta^2)}, \quad (57)$$

$$\psi(\eta) = (1+\nu) \alpha a^2 \sum_{m=1}^{\infty} \frac{Q_m J_1(j_m)}{j_m} \left\{ \frac{1 - \cos \eta}{\eta} - \frac{1 - J_0(\eta)}{\eta} + \frac{\eta J_0(\eta)}{j_m^2 - \eta^2} \right\}. \quad (58)$$

If we make use of equation (26) and the results

$$\int_0^{\infty} \frac{J_0(\eta) - \cos \eta}{\eta} J_0(\varrho \eta) d\eta = \begin{cases} \cosh^{-1}(1/\varrho) - \log(1/\varrho), & 0 < \varrho < 1, \\ 0, & \varrho > 1, \end{cases}$$

$$\int_0^{\infty} \frac{\eta J_0(\eta) J_0(\varrho \eta) d\eta}{j_m^2 - \eta^2} = \frac{J_0(j_m \varrho)}{j_m J_1(j_m)},$$

we see that the normal component of the surface displacement is given by the equation

$$w_0 = (1+\nu) \alpha a^2 \left\{ \log[1 + \sqrt{1 - \varrho^2}] \sum_{m=1}^{\infty} \frac{Q_m J_1(j_m)}{j_m} + \sum_{m=1}^{\infty} \frac{Q_m J_0(j_m \varrho)}{j_m^2} \right\}. \quad (59)$$

If we add equations (57) and (58), we find that

$$\psi(\eta) + \varphi(\eta) = -(1+\nu) \alpha a^2 \sum_{m=1}^{\infty} \frac{Q_m}{j_m} \cdot \frac{J_1(j_m)}{\eta} \cos \eta. \quad (60)$$

Substituting this expression into the right-hand side of equation (31), we find that the normal component of stress is given by the equation

$$\sigma_z = \frac{\mu(1+\nu) \alpha a}{1-\nu} C f(\varrho, \zeta), \quad (61)$$

where

$$C = \sum_{m=1}^{\infty} \frac{Q_m J_1(j_m)}{j_m} \quad (62)$$

is a number which is independent of ϱ and ζ and the function $f(\varrho, \zeta)$ is defined by equation (52) above.

If we multiply both sides of equation (54) by ϱ and integrate with respect to ϱ from 0 to ϱ , we find that

$$\frac{1}{\varrho} \int_0^{\varrho} s Q(s) ds = \sum_{m=1}^{\infty} \frac{Q_m}{j_m} J_1(j_m \varrho) \quad (63)$$

so that, putting $\varrho = 1$, we find immediately that

$$C = \int_0^1 s Q(s) ds. \quad (64)$$

The striking fact proved by equation (61) is that, apart from the value of the multiplicative constant C , the form of the variation of σ_z with r and z is

independent of the values of the constants $Q_1, Q_2, \dots$ occurring in the Fourier-Bessel expansion of the flux function $Q(r/a)$. This rather special behaviour is reminiscent of the degeneracy noted by STERNBERG & McDOWELL (1957) in the analysis of the thermal stresses in a semi-infinite elastic solid. A similar result does not hold for the components of the displacement vector. For instance, equation (59) shows that the form of w_0 depends upon the values of the Fourier-Bessel coefficients $Q_1, Q_2, \dots$. We can, in fact, show the relation between w_0 and $Q(r/a)$ more clearly. If we integrate both sides of equation (64) we obtain the relation

$$\int_{\varrho}^1 \frac{dt}{t} \int_0^t s Q(s) ds = \sum_{m=1}^{\infty} \frac{Q_m J_0(j_m \varrho)}{j_m^2}, \quad 0 < \varrho < 1. \quad (65)$$

If we substitute this expression into equation (59), we obtain the equation

$$w_0 = (1 + \nu) \alpha a^2 \left\{ C \log [1 + \sqrt{1 - \varrho^2}] + \int_{\varrho}^1 \frac{dt}{t} \int_0^t s Q(s) ds \right\}, \quad 0 < \varrho < 1 \quad (66)$$

where the constant C is defined by equation (64).

For flux-functions of the type

$$Q(\varrho) = c_0 + c_1 \varrho^2 + c_2 \varrho^4 + \dots \quad (67)$$

we therefore find that

$$w_0 = (1 + \nu) \alpha a^2 \{ C \log [1 + \sqrt{1 - \varrho^2}] + W(\varrho) \}, \quad 0 < \varrho < 1, \quad (68)$$

where

$$W(\varrho) = \frac{1}{4} \sum_{r=0}^{\infty} \frac{c_r}{(r+1)^2} (1 - \varrho^{2r+2}) \quad (69)$$

and

$$C = \frac{1}{2} \sum_{r=0}^{\infty} \frac{c_r}{r+1}. \quad (70)$$

For example:

(a) $Q(\varrho) = Q_0$, a constant.

If the flux function is a constant then it is easily shown from equation (68) that the normal component of the surface displacement is given by equation (48) above.

(b) $Q(\varrho) = Q_0(1 - \varrho^2)$.

For this parabolic flux function we have the expression

$$w_0 = \frac{1}{4} (1 + \nu) \alpha a^2 Q_0 \left\{ \log [1 + \sqrt{1 - \varrho^2}] + (1 - \varrho^2) - \frac{1}{4} (1 - \varrho^4) \right\} \quad (71)$$

for the normal component of the surface displacement.

(c) $Q(\varrho) = Q_0(1 - \varrho^4)$.

For this flux function we obtain the equation

$$w_0 = \frac{1}{4} (1 + \nu) \alpha a^2 Q_0 \left\{ \frac{4}{3} \log [1 + \sqrt{1 - \varrho^2}] + (1 - \varrho^2) - \frac{1}{9} (1 - \varrho^6) \right\}. \quad (72)$$

(d) $Q(\varrho) = Q_0(1 - \varrho^2)^2$.

In this case we obtain the expression

$$w_0 = \frac{1}{4}(1+\nu) \alpha a^2 Q_0 \left\{ \frac{2}{3} \log[1 + \sqrt{1 - \varrho^2}] + (1 - \varrho^2) - \frac{1}{2}(1 - \varrho^4) + \frac{1}{9}(1 - \varrho^6) \right\} \quad (73)$$

for the normal component of the surface displacement.

The variation of the normal component of the surface displacement for these four forms of the flux function is shown in Fig. 2. In this diagram, the length L is defined by the equation

$$L = \frac{1}{4}(1+\nu) \alpha a^2 Q_0. \quad (74)$$

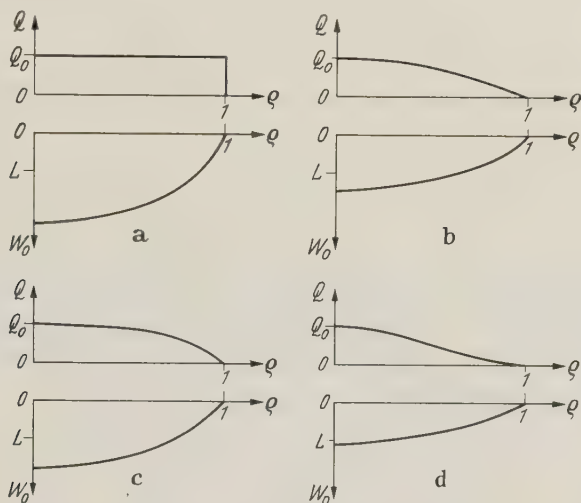


Fig. 2. The normal component of the surface displacement in the four cases considered in § 5. In each case the upper curve shows the flux function and the lower curve the displacement component

6. Prescribed Temperature at the Surfaces of a Penny-Shaped Crack

In this section we shall discuss briefly the case in which a penny-shaped crack is opened out by the application of a prescribed temperature to its flat surfaces. In this case the boundary conditions (35) and (36) remain unaltered but equations (33) and (34) are replaced by the pair of equations

$$\vartheta = -\vartheta_0 g(r/a), \quad 0 \leq r \leq a; \quad (75)$$

$$\frac{\partial \vartheta}{\partial z} = 0, \quad r > a. \quad (76)$$

In equation (75) ϑ_0 is a constant and $g(r/a)$ is a prescribed function of r/a .

It is easily shown that the solution of Laplace's equation (3) satisfying these thermal boundary conditions is

$$\vartheta = -a \int_0^\infty q(\eta) J_0(\varrho \eta) e^{-\zeta \eta} d\eta \quad (77)$$

with

$$\frac{\partial \vartheta}{\partial z} = \int_0^\infty \eta q(\eta) J_0(\varrho \eta) e^{-\eta \zeta} d\eta. \quad (78)$$

For this solution, therefore,

$$Q(\varrho) = \int_0^{\infty} \eta q(\eta) J_0(\varrho \eta) d\eta, \quad 0 < \varrho < 1. \quad (79)$$

As a result of the boundary conditions (75) and (76) we see that the function $q(\eta)$ must be taken to be the solution of the dual integral equations

$$\int_0^{\infty} q(\eta) J_0(\varrho \eta) d\eta = \frac{\vartheta_0}{a} g(\varrho), \quad 0 < \varrho < 1, \quad (80)$$

$$\int_0^{\infty} \eta q(\eta) J_0(\varrho \eta) d\eta = 0, \quad \varrho > 1. \quad (81)$$

The solution of these equations is known to be

$$q(\eta) = \frac{2\vartheta_0}{\pi a} \cos \eta \int_0^1 \frac{y g(y) dy}{\sqrt{1-y^2}} + \frac{2\vartheta_0}{\pi a} \eta \int_0^1 \frac{y dy}{1-y^2} \int_0^1 u g(yu) \sin(\eta u) du \quad (82)$$

(SNEDDON 1951, p. 65). If we substitute from equation (82) into equation (79) we obtain the expression for $Q(\varrho)$, the flux of heat across the surfaces of the crack. Once we have found this function, we can proceed using the method of § 4 above. The following results are useful in applying this method.

From equations (41) and (79) we find that

$$\mathcal{Q}(\varrho) = \int_0^{\infty} q(\eta) [J_1(\eta) - \varrho J_1(\eta \varrho)] d\eta, \quad (83)$$

and hence that

$$\mathcal{Q}(0) = \int_0^{\infty} q(\eta) J_1(\eta) d\eta. \quad (84)$$

Similarly, from equations (64) and (79), we find that

$$C = \int_0^{\infty} q(\eta) J_1(\eta) d\eta, \quad (85)$$

and, from equation (79) that

$$W(\varrho) = \int_{\varrho}^1 \frac{dt}{t} \int_0^t s Q(s) ds = \int_0^{\infty} \frac{q(\eta)}{\eta} [J_0(\varrho \eta) - J_0(\eta)] d\eta. \quad (86)$$

For instance, if we substitute from equations (85) and (86) into equation (68) we obtain the appropriate expression for the normal component of the displacement of the surface of the crack, and if we substitute from equation (85) into equation (61) we get the expression for the normal component of stress at any point in the solid.

To illustrate the procedure, we consider the simple case given by the equations

$$\begin{aligned} \vartheta &= -\vartheta_0, & 0 \leq r \leq a, \quad z = 0; \\ \frac{\partial \vartheta}{\partial z} &= 0, & r > a, \quad z = 0. \end{aligned}$$

In this case $g(r/a) = 1$ and it is easily found that

$$q(\eta) = \frac{2\vartheta_0}{\pi a} \frac{\sin \eta}{\eta}$$

so that

$$Q(\varrho) = \frac{2\vartheta_0}{\pi a} \int_0^\infty \sin \eta J_0(\varrho \eta) d\eta.$$

From this expression it is readily verified that $Q(\varrho) = 0$ if $\varrho > 1$, and that

$$Q(\varrho) = \frac{2\vartheta_0}{\pi a} (1 - \varrho^2)^{-\frac{1}{2}}, \quad 0 < \varrho < 1.$$

By the definition of $\mathcal{Q}(\varrho)$, we find that

$$\mathcal{Q}(\varrho) = \frac{2\vartheta_0}{\pi a} (1 - \varrho^2)^{\frac{1}{2}}.$$

Equation (42) then yields the expression

$$\psi(\eta) = \frac{2(1+\nu)\alpha\vartheta_0 a}{\pi} \left\{ \frac{1 - \cos \eta}{\eta} - \int_0^\infty \sqrt{1 - \varrho^2} J_1(\varrho \eta) d\varrho \right\} \quad (87)$$

for the function $\psi(\eta)$. Evaluating the integral we find that, for this problem,

$$\psi(\eta) = \frac{(1+\nu)\alpha\vartheta_0 a}{\pi \eta} \left(2 - \cos \eta - \frac{\sin \eta}{\eta} \right). \quad (88)$$

Similarly, equation (37) gives the equation

$$\varphi(\eta) = - \frac{2(1+\nu)\alpha\vartheta_0 a}{\pi \eta} \cdot \frac{\sin \eta}{\eta}. \quad (89)$$

The solution of the problem is then obtained by substituting from equations (88) and (89) into equations (22), (23) and (24).

For this problem

$$\int_0^t s Q(s) ds = \frac{2\vartheta_0}{3\pi a} [1 - (1 - t^2)^{\frac{3}{2}}],$$

so that

$$C = \frac{2\vartheta_0}{3\pi a}$$

and

$$\begin{aligned} W(\varrho) &= \frac{2\vartheta_0}{3\pi a} \int_0^1 \frac{dt}{t} [1 - (1 - t^2)^{\frac{3}{2}}] \\ &= \frac{2\vartheta_0}{3\pi a} \left[(1 - \varrho^2)^{\frac{1}{2}} + \frac{1}{3} (1 - \varrho^2)^{\frac{3}{2}} - \log [1 + \sqrt{1 - \varrho^2}] \right]. \end{aligned}$$

Substituting these expressions into equation (68), we find that, in this case, the normal component of the displacement of the crack surface is given by the equation

$$w_0 = L_0 \left[(1 - \varrho^2)^{\frac{1}{2}} + \frac{1}{3} (1 - \varrho^2)^{\frac{3}{2}} \right] \quad (90)$$

where

$$L_0 = \frac{2(1+\nu)\alpha\vartheta_0 a}{3\pi}. \quad (91)$$

In a similar way we obtain the expression

$$\sigma_z = \frac{2(1+\nu)\alpha\vartheta_0 a}{3\pi(1-\nu)} \mu f(\varrho, \zeta) \quad (92)$$

for the normal component of stress at any point in the solid.

The variation of w_0 with ϱ is illustrated in Fig. 3; that of σ_z can be inferred from Fig. 1.

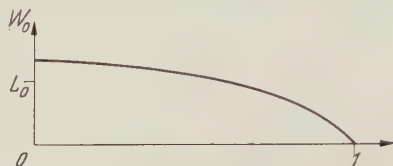


Fig. 3. The normal component of the surface displacement when the surface of the crack is maintained at a temperature ϑ_0 below the temperature of the reference state

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References

- GOODIER, J. N.: *Surveys in Applied Mathematics I: The Mathematical Theory of Elasticity*. New York: Wiley 1958.
- IRWIN, G. R.: *Fracture. Handbuch der Physik*, Bd. VI. Berlin-Göttingen-Heidelberg: Springer 1958.
- SNEDDON, I. N.: *Proc. Roy. Soc. A* **187**, 229–260 (1946).
- SNEDDON, I. N.: *Fourier Transforms*. New York: McGraw-Hill 1951.
- SNEDDON, I. N., & D. S. BERRY: *The Classical Theory of Elasticity. Handbuch der Physik*, Bd. VI. Berlin-Göttingen-Heidelberg: Springer 1958.
- SNEDDON, I. N., & F. J. LOCKETT: *Quarterly of Applied Mathematics*. 1960 (in course of publication).
- STERNBERG, E., & McDOWELL: *Quart. App. Math.* **14**, 381 (1957).

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Null Solutions of Partial Differential Equations

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1. Introduction

Let $P(D) = P(-i\partial/\partial x^1, \dots, -i\partial/\partial x^n)$ be a partial differential operator with constant coefficients. A solution $u \in \mathcal{E}(R^n)^*$ of the equation

$$(1.1) \quad P(D)u = 0$$

which vanishes outside the half space $H = \{x; \langle x, N \rangle > 0\}$ will be called a null solution of $P(D)$ with respect to H . It was proved by HÖRMANDER [I], Theorem 3.2 that there is a null solution $\neq 0$ with respect to H if and only if the bounding hyperplane $\langle x, N \rangle = 0$ is characteristic, that is,

$$(1.2) \quad P^0(N) = 0$$

where P^0 is the principal part of the polynomial P . Our aim here is to investigate when all solutions can be approximated by null solutions.

Theorem 1. *Let $P(\zeta) = P_1(\zeta)P_2(\zeta)$ where the plane $\langle x, N \rangle = 0$ is characteristic with respect to every irreducible factor of P_1 but non-characteristic with respect to P_2 . Let H be the half space $\{x; \langle x, N \rangle > 0\}$. Then the closure in $\mathcal{E}(H)$ of the null solutions of $P(D)$ with respect to H is the set of all solutions in $\mathcal{E}(H)$ of the equation $P_1(D)u = 0$.*

In proving this theorem we may assume that $P_2 = 1$. For if

$$P(D)u = P_2(D)(P_1(D)u) = 0$$

and u vanishes outside H , the function $P_1(D)u$ is a null solution for $P_2(D)$, hence identically zero, since the plane $\langle x, N \rangle = 0$ is non-characteristic for $P_2(D)$. Hence $P_1(D)u = 0$, which shows that the null solutions of $P(D)$ and of $P_1(D)$ are the same. Thus we may assume in the proof that the plane $\langle x, N \rangle = 0$ is characteristic with respect to every irreducible factor of $P(\zeta)$.

A weaker result than Theorem 1 is the following

Theorem 2. *Let P fulfill the same assumptions as in Theorem 1. Then the closure in $\mathcal{E}(R^n)$ of the set of functions u which are null solutions with respect to the half space $\{x; \langle x, N \rangle > c\}$, for some c , consists of all solutions in $\mathcal{E}(R^n)$ of the equation $P_1(D)u = 0$.*

* By $\mathcal{E}(\Omega)$, where Ω is an open set, we denote the set of infinitely differentiable functions in Ω with the Schwartz topology of uniform convergence on every compact set of the derivatives of all orders, including 0. For the definition of $\mathcal{E}'(\Omega)$ and $\mathcal{D}(\Omega)$ we refer to SCHWARTZ [5].

2. Preliminary lemmas

Our first lemma is of an algebraic nature:

Lemma 1. *Let $Q(\zeta)$ be an irreducible polynomial such that $Q(sN + \xi)$ is not independent of s for indeterminate s and ξ . Then the polynomial*

$$Q(sN + t\xi + \xi_0)$$

is an irreducible polynomial of s and t except when (ξ, ξ_0) belongs to a certain algebraic set. Hence there is an algebraic set A such that when $\xi \notin A$ the polynomial is irreducible except when ξ_0 belongs to an algebraic set, depending on ξ .

Here we mean by an algebraic set a set defined by algebraic equations and different from the whole space.

The author has been informed by Professor P. SAMUEL that this result is well known though not explicitly stated in the literature. A proof may be based on pp. 211–218 in LANG [3] but will not be given here. A somewhat weaker lemma was used in a similar context by JOHN [3, pp. 397–398].

Lemma 2. *Let $F(\zeta)$ be an entire analytic function and $P(\zeta)$ a polynomial. Suppose that there is a vector ξ such that $P^0(\xi) \neq 0$ and*

$$F(t\xi + \xi_0)/P(t\xi + \xi_0)$$

is an analytic function of t for a dense set of complex ξ_0 . Then $F(\zeta)/P(\zeta)$ is an entire analytic function.

Proof. Since the coefficient in $P(t\xi + \xi_0)$ of the highest power of t is independent of ξ_0 in view of the fact that $P^0(\xi) \neq 0$, the zeros t of the equation $P(t\xi + \xi_0) = 0$ vary continuously with ξ_0 . For all ξ_0 in a dense set the zeros of $P(t\xi + \xi_0)$ are by hypothesis zeros of $F(t\xi + \xi_0)$ with at least the same multiplicity. Hence it immediately follows that this is true for all ξ_0 .

We can now define $G(\xi_0) = \lim_{t \rightarrow 0} F(t\xi + \xi_0)/P(t\xi + \xi_0)$; the limit exists because the quotient is an analytic function of t , and $G(\xi_0) = F(\xi_0)/P(\xi_0)$ if $P(\xi_0) \neq 0$. Choose r so that

$$P(t\xi + \xi_0) \neq 0, \quad |t| = r.$$

Now we have by CAUCHY'S integral formula

$$G(\zeta) = (2\pi i)^{-1} \int_{|t|=r} \frac{F(t\xi + \zeta)}{P(t\xi + \zeta)} \frac{dt}{t}.$$

For ζ in a neighbourhood of ξ_0 we have $P(t\xi + \zeta) \neq 0$ in the range of integration. Hence the integral is there an analytic function of ζ , and $G(\zeta)$ is thus an entire function.

We finally give a lemma on Fourier transforms, which is also well known. The Fourier transform of an integrable function φ is here defined by

$$\hat{\varphi}(\zeta) = \int e^{i\langle x, \zeta \rangle} \varphi(x) dx.$$

Lemma 3. *Let $\delta > 1$ and φ be a function with compact support such that with a constant C*

$$(2.1) \quad |D^\alpha \varphi(x)| \leq C^{|\alpha|+1} (|\alpha|!)^\delta.$$

Then the Fourier transform has the estimate

$$(2.2) \quad |\hat{\varphi}(\xi + i\eta)| \leq C_1 \exp(A|\eta| - B|\xi|^{1/\delta}),$$

where C_1, A, B are constants > 0 .

Proof. From (2.1) it follows with a constant C_2 that

$$(2.3) \quad |\zeta|^k |\hat{\varphi}(\zeta)| \leq C_2^{k+1} (k!)^\delta e^{A|\operatorname{Im} \zeta|}, \quad \zeta = \xi + i\eta,$$

which gives

$$(2.3)' \quad |\hat{\varphi}(\xi + i\eta)| \leq C_2 \left(\frac{C_2}{|\xi|} \right)^k (k!)^\delta e^{A|\eta|}.$$

In order to make the right-hand side of (2.3)' as small as possible we choose k as the largest integer $< (|\xi|/C_2)^{1/\delta}$. Using STIRLING's formula

$$k! = \left(\frac{k}{e} \right)^k \sqrt{2\pi k} (1 + o(1))$$

we then obtain (2.2).

3. Proof of Theorem 1

We have to prove that every distribution $T \in \mathcal{E}'(H)$ which is orthogonal to every null solution with respect to H is in fact orthogonal to every solution of (1.1) in $\mathcal{E}(H)$. We first reduce our study to such distributions T as are very smooth functions; this is essential for the convergence of the integrals occurring below.

Thus let ϑ be a function in $\mathcal{D}(-H)$ for which (2.1) holds and $\int \vartheta dx = 1$. Such a function exists for every $\delta > 1$. Indeed, it is well known that the function which equals 0 for $x_1 \leq 0$ and $\exp(-x_1^{1-\delta})$ for $x_1 > 0$ satisfies (2.1). Starting from this function, we easily construct a function ϑ with the required properties, since the set of functions satisfying (2.1) for some C is a ring (cf. SCHWARTZ [5, p. 22]). Now form $T_\varepsilon = T * \vartheta_\varepsilon$ where $\vartheta_\varepsilon(x) = \varepsilon^{-n} \vartheta(x/\varepsilon)$, $\varepsilon > 0$. When $\varepsilon \rightarrow 0$ we have $T_\varepsilon \rightarrow T$ in the topology of $\mathcal{E}'(H)$; and $T_\varepsilon \in \mathcal{D}(H)$ when ε is small enough. Moreover, T_ε satisfies the condition (2.1) for every ε , hence $\hat{T}_\varepsilon$ satisfies (2.2).

Now it is easy to see that if T is orthogonal to all null solutions u with respect to H , then so is T_ε . For

$$T_\varepsilon(u) = T_\varepsilon * \check{u}(0) = T * \vartheta_\varepsilon * \check{u}(0) = T(\check{\vartheta}_\varepsilon * u),$$

where $\check{u}(x) = u(-x)$. It is clear that $\check{\vartheta}_\varepsilon * u$ is also a null solution with respect to H , hence $T_\varepsilon(u) = 0$. If we can prove that T_ε is orthogonal to all solutions $u \in \mathcal{E}(H)$ of (1.1), it will follow when $\varepsilon \rightarrow 0$ that

$$T(u) = \lim_{\varepsilon \rightarrow 0} T_\varepsilon(u) = 0.$$

Hence it is sufficient to show for some $\delta > 1$ that if a function $\varphi \in \mathcal{D}(H)$ satisfying (2.1) is orthogonal to all null solutions, then φ is orthogonal to all solutions.

We now recall the construction of null solutions given by HÖRMANDER [I], giving it a slightly more general form. Let

$$P(\zeta) = Q_1(\zeta)^{\mu_1} \dots Q_r(\zeta)^{\mu_r}$$

be the decomposition of P in irreducible factors, $Q_1, \dots, Q_r$ being different. We know by hypothesis (see Section 1) that

$$(3.1) \quad Q_j^0(N) = 0, \quad j = 1, \dots, r.$$

Let ξ and ξ_0 be fixed vectors such that ξ is real and

$$(3.2) \quad Q_j^0(\xi) \neq 0, \quad j = 1, \dots, r.$$

It is then easy to see that the equation

$$(3.3) \quad Q_j(sN + t\xi + \xi_0) = 0$$

has a solution $t(s)$ with a Puiseux series expansion

$$t = s^{k/p} \sum_0^{\infty} c_j s^{-j/p}$$

convergent when $|s| \geq M$. (Cf. VAN DER WAERDEN [6, pp. 50–52].) Furthermore, which is the important point, we have $k < p$ so that

$$|t(s)| = O(|s|^{1-1/p}), \quad s \rightarrow \infty.$$

Let ϱ be a number such that

$$1 - 1/p < \varrho < 1,$$

and set with $\varepsilon > 0$

$$(3.4) \quad u(x) = \int_{C^-} e^{i\langle x, sN + t(s)\xi + \xi_0 \rangle} s^{v/p} e^{-\varepsilon(is)^{\varrho}} ds.$$

Here we define $(is)^{\varrho}$ so that it is real and positive when s is on the negative imaginary axis, and we choose a fixed branch of $s^{1/p}$ in the lower half plane. The contour C^- is composed by the parts of the real axis where $|s| \geq M$ and the half of the circle $|s| = M$ in the lower half plane. The curve is described from $-\infty$ to $+\infty$. Finally v is an arbitrary integer.

Arguing precisely as on p. 217 in HÖRMANDER [I] we find that u is a null solution of $Q_j(D)$ with respect to H . In fact, u is a solution because of (3.3), and a null solution because in (3.4) the integration contour can be deformed to the line $\text{Im } s = \tau$ if $\tau < -M$; when $\tau \rightarrow -\infty$ an easy estimation shows that u vanishes when $\langle x, N \rangle < 0$. The details of the proof need not be repeated here. Since $Q_j(D)$ is a factor of $P(D)$ of order μ_j it follows from LEIBNIZ' formula that $R(x)u(x)$ is a null solution of $P(D)$ if $R(x)$ is a polynomial of degree $< \mu_j$.

Now let $\varphi \in \mathcal{D}(H)$ be orthogonal to all null solutions with respect to H . Differentiating the equation

$$\hat{\varphi}(\xi) = \int e^{i\langle x, \xi \rangle} \varphi(x) dx,$$

we obtain

$$R(D) \hat{\varphi}(\xi) = \int e^{i\langle x, \xi \rangle} R(x) \varphi(x) dx$$

if R is a polynomial. Thus if we multiply (3.4) by $R(x)\varphi(x)$ and integrate with respect to x , if the degree of R is $< \mu_j$, so that $R(x)u(x)$ is a null solution, we get

$$(3.5) \quad \int_{C^-} (R(D) \hat{\varphi})(sN + t(s)\xi + \xi_0) s^{v/p} e^{-\varepsilon(is)^{\varrho}} ds = 0.$$

The change of the order of integrations is legitimate because we have an absolutely convergent double integral.

We want to let $\varepsilon \rightarrow 0$ in (3.5). In order to be able to do so we have only to make sure that the integral is absolutely convergent for $\varepsilon = 0$. For the factor $\exp(-\varepsilon(is)^q)$ is < 1 in absolute value on C^- . Now, as pointed out above, we may assume that $R(D)\hat{\varphi}$ satisfies an inequality of the form (2.2). Hence with suitable constants C_1, A, B

$$\begin{aligned} & |(R(D)\hat{\varphi})(sN + t(s)\xi + \xi_0)| \\ & \leq C_1 \exp(A|N \operatorname{Im} s + \xi \operatorname{Im} t(s) + \operatorname{Im} \xi_0| - B|N \operatorname{Re} s + \xi \operatorname{Re} t(s) + \operatorname{Re} \xi_0|^{1/\delta}). \end{aligned}$$

Since $|t(s)| = o(|s|^{1-1/p})$, it will follow that the integral (3.5) with $\varepsilon = 0$ is absolutely convergent if

$$1 - \frac{1}{p} < \frac{1}{\delta}, \quad \text{that is, if } \delta < p/(p-1).$$

Let m be the degree of P and take δ so that

$$1 < \delta < m/(m-1).$$

Since $p \leq m$ we then have $\delta < p/(p-1)$ so that the integral (3.5) is absolutely convergent when $\varepsilon = 0$. Letting $\varepsilon \rightarrow 0$ we thus get, writing

$$(R(D)\hat{\varphi})(sN + t(s)\xi + \xi_0) = F(s^{1/p})$$

so that F is a univalent analytic function,

$$(3.6) \quad \int_{C^-} F(s^{1/p}) s^{v/p} ds = 0.$$

On the other hand, if in (3.4) we integrate over the curve C^+ which is symmetric to C^- with respect to the real axis and is described from $+\infty$ to $-\infty$ and replace the convergence factor $e^{-\varepsilon(s/i)^q}$ by the factor $e^{-\varepsilon(s i)^q}$, we will get a null solution u with respect to $-H$. Hence u vanishes in the support of φ , and is thus orthogonal to $R\varphi$. Repeating the same argument as before, we obtain

$$(3.7) \quad \int_{C^+} F(s^{1/p}) s^{v/p} ds = 0.$$

The equations (3.7) and (3.6) are valid for all the different determinations of $s^{1/p}$. Adding the equalities (3.6) and (3.7) for all the determinations, we get

$$(3.8) \quad \int F(s^{1/p}) s^{v/p} ds = 0$$

where the integral is taken p times round the circle $|s| = M$. In fact, it is obvious that the integrals along the real axis cancel each other. Setting $s = z^p$, we get

$$(3.9) \quad \int_{|z|=M^{1/p}} F(z) z^{v+p-1} dz = 0$$

for all integers v . Hence all coefficients in the Laurent series for $F(z)$ are zero, so that $F(z) = 0$ identically.

Before proceeding we have to choose ξ in a suitable way; up to now our only conditions on ξ have been the conditions (3.2). We now also demand that for those $Q_j, j = 1, \dots, r'$, which are not constant in the direction N , the vector ξ shall have the property given by Lemma 1. These requirements can be fulfilled

at the same time as (3.2) since each condition means only that ξ shall not belong to a certain algebraic set. Thus we now know that the polynomials $Q_j(sN + t\xi + \xi_0)$, $j=1, \dots, r'$, are irreducible if $\xi_0 \notin A$, where A is also an algebraic set.

When $\xi_0 \notin A$ it follows if $j=1, \dots, r'$, that $(R(D)\hat{\varphi})(sN + t\xi + \xi_0) = 0$ if the degree of R is $< \mu_j$ and (s, t) is on the Riemann surface of the algebraic function defined by

$$Q_j(sN + t\xi + \xi_0) = 0.$$

In fact, this surface is *connected* because of the irreducibility, and we have proved previously that the assertion is true in a neighbourhood of infinity in a certain sheet. Hence it follows in general by analytic continuation. Hence $\hat{\varphi}(sN + t\xi + \xi_0)$ will as a function of t have a μ_j -fold zero wherever $Q_j(sN + t\xi + \xi_0)$ has a zero, and passing to the limit from such s for which the zeros are simple, we also find that a zero of Q_j of multiplicity k will be a zero of $\hat{\varphi}$ of multiplicity $k\mu_j$ at least. In particular, for $s=0$ we find that

$$\hat{\varphi}(t\xi + \xi_0)/Q_j(t\xi + \xi_0)^{\mu_j}$$

is an analytic function of t when $\xi_0 \notin A$.

If $j=r'+1, \dots, r$ so that Q_j is constant in the direction N , it is even more easy to get the same result. In fact, if $Q_j(t\xi + \xi_0)$, $j > r'$ has a zero t_0 of order μ , then

$$u(x) = \langle x, \xi \rangle^k e^{i\langle x, t_0\xi + \xi_0 \rangle}$$

is a solution of the equation $Q_j(D)^{\mu_j} u = 0$ if $k < \mu_j \mu$. We can find a function $\chi \in C_0^\infty(0, \infty)$ so that $\chi(\langle x, N \rangle) = 1$ in a neighbourhood of the support of φ . Since Q_j is constant in the direction N we have

$$Q_j(D)^{\mu_j}(\chi u) = \chi Q_j(D)^{\mu_j} u = 0.$$

Hence χu is a null solution, so that

$$\int \varphi u dx = \int \varphi(\chi u) dx = 0.$$

It follows that $\hat{\varphi}(t\xi + \xi_0)$ has a zero of order $\mu_j \mu$ at least for $t=t_0$. This proves that

$$\hat{\varphi}(t\xi + \xi_0)/Q_j(t\xi + \xi_0)^{\mu_j}$$

is an analytic function of t .

Now let B be the set of all ξ_0 such that at least two of the polynomials $Q_j(t\xi + \xi_0)$, $j=1, \dots, r$, considered as polynomials in t have a common zero. B is an algebraic set since it is the set where the resultant of Q_j and Q_k vanishes for some pair $j \neq k$ and none of these resultants is $\equiv 0$. When $\xi_0 \notin (A \cup B)$ we now get that

$$\hat{\varphi}(t\xi + \xi_0)/P(t\xi + \xi_0) = \hat{\varphi}(t\xi + \xi_0) / \prod_1^r Q_j(t\xi + \xi_0)^{\mu_j}$$

is analytic. In virtue of Lemma 2 it follows that $\hat{\varphi}(\zeta)/P(\zeta)$ is an entire function. Hence according to MALGRANGE [4, Chap. I, Prop. 2], we have $\hat{\varphi}(\zeta)/P(\zeta) = \hat{\psi}(\zeta)$ where $\psi \in \mathcal{E}'(H)$ (in fact, $\psi \in \mathcal{D}(H)$). From the equation $\hat{\varphi}(\zeta) = P(\zeta)\hat{\psi}(\zeta)$ it follows that

$$\varphi = P(-D)\psi.$$

Hence if $u \in \mathcal{E}(H)$ and $P(D)u=0$, we get

$$\int u \varphi dx = (P(-D)\psi)(u) = \psi(P(D)u) = 0,$$

which completes the proof of Theorem 1.

4. Proof of Theorem 2

Again, what we have to prove is that if φ is orthogonal to all null solutions, then φ is orthogonal to all solutions of (1.1). But if we take the half space H so that φ has compact support in H , it follows from what we have already proved that $\varphi = P(-D)\psi$ where ψ has compact support in H . Hence φ is orthogonal to all $u \in \mathcal{E}(R^n)$ with $P(D)u=0$, which proves the statement.

References

- [1] HÖRMANDER, L.: On the theory of general partial differential operators. Acta Math. **94**, 161–248 (1955).
- [2] JOHN, F.: Non-admissible data for differential equations with constant coefficients. Comm. Pure Appl. Math. **10**, 391–398 (1957).
- [3] LANG, S.: Introduction to algebraic geometry. New York 1958.
- [4] MALGRANGE, B.: Existence et approximation des solutions des équations aux dérivées partielles et les équations de convolution. Ann. Inst. Fourier Grenoble **6**, 271–355 (1956).
- [5] SCHWARTZ, L.: Théorie des distributions, Tome I. Paris 1950.
- [6] VAN DER WAERDEN, B. L.: Einführung in die algebraische Geometrie. Leipzig 1939.

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The Formulation of Constitutive Equations in Continuum Physics. II

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1. Introduction

In previous papers by GREEN & RIVLIN [1, 2] and an earlier report [3], constitutive equations have been developed describing the mechanical properties of materials possessing memory. A similar topic has been discussed by NOLL [4, 5]. In [1, 2] it is assumed that the stress at a fixed particle of the material is a functional of the displacement gradients at the instant of measurement and at times preceding the instant of measurement. In another paper, by PIPKIN & RIVLIN [6], constitutive equations have been developed governing certain types of physical properties of materials which may be described by the relation between a tensor of order μ on the one hand and the displacement gradients in the material and ν vectors on the other, all the quantities being measured simultaneously. In the present paper, these two problems are combined.

We are concerned with physical properties of materials which can be described in a rectangular Cartesian coordinate system by the relation between the components at some instant t of a tensor $\mathbf{u}$ of order μ on the one hand and the components of ν vectors and the displacement gradients on the other, at all times from some fixed initial time up to and including the time t . The components of $\mathbf{u}$ are therefore functionals of the ν vectors and of the displacement gradients over a time interval preceding t .

The limitations imposed on the constitutive equation by the assumption that it is unaltered by a simultaneous rotation of the reference system and of the physical system are derived. The manner in which we may derive the further restrictions due to the symmetry of the material is discussed and illustrated in the particular case when the material is isotropic and possesses a center of symmetry.

Finally, in §§6 and 7, we discuss the manner in which these constitutive equations may be considerably simplified if we restrict ourselves to certain classes of deformations in which the displacement vector and ν independent vectors in the constitutive equation each vary in a prescribed fashion with time. §6 deals with the case in which the body is undeformed and §7 with the case in which the displacement gradients are small compared with unity. The approach taken is similar to that adopted in a previous paper [7] in which is discussed the constitutive equation appropriate to purely mechanical problems in which the stress at any instant of time depends on the displacement gradients at and preceding that instant, the displacement gradients are small compared with unity and the displacement vector varies in a prescribed manner with time.

2. Restrictions on the constitutive equation due to rotation in implicit form

We consider a body to undergo deformation such that a generic particle of the body which at time $\tau=0$ is at X_i in the rectangular Cartesian coordinate system x moves to x_i in the same system at time τ . Then, the deformation of the body is described by the dependence of x_i on X_j and τ , thus:

$$x_i = x_i(X_j, \tau). \quad (2.1)$$

We assume that some physical property of the material is described by the dependence of the components $u_{i_1 i_2 \dots i_\mu}(t)$ of a tensor of order μ at time t on the values of the deformation gradients $\partial x_p(\tau)/\partial X_q$ and of the components in the system x of ν vectors $v_p^{(\alpha)}(\tau)$, ($\alpha=1, 2, \dots, \nu$), at all times in the interval 0 to τ . This may be expressed mathematically by

$$u_{i_1 i_2 \dots i_\mu} = F_{i_1 i_2 \dots i_\mu} \left[\partial x_p(\tau) / \partial X_q, v_p^{(\alpha)}(\tau) \right], \quad (2.2)$$

where $F_{i_1 i_2 \dots i_\mu}$ denotes functional dependence on the arguments. In (2.2), we employ the notation

$$u_{i_1 i_2 \dots i_\mu} = u_{i_1 i_2 \dots i_\mu}(t). \quad (2.3)$$

The assumption will be made later that $F_{i_1 i_2 \dots i_\mu}$ is a continuous functional of its arguments in the interval 0 to τ and is particularly dependent on the values of the arguments at $\tau=t$.

We assume also that the related physical variables are such that the constitutive equation (2.2) is unaltered by a rotation of the physical system as a whole, if it is expressed in a coordinate system which is subjected to the same rotation. The restrictions which are thereby imposed on the constitutive equation (2.2) may be obtained in the following manner.

Let us assume that in addition to the deformation described by (2.1), the body is subjected, subsequently to time $\tau=0$, to a time-dependent rigid rotation in which a point at $x_i(\tau)$ in the system x is moved to $\bar{x}_i(\tau)$ in the same system, where

$$\bar{x}_i(\tau) = a_{ij}(\tau) x_j(\tau). \quad (2.4)$$

The quantities $a_{ij}(\tau)$ satisfy, of course, the orthogonality conditions

$$a_{ij}(\tau) a_{ik}(\tau) = a_{ji}(\tau) a_{ki}(\tau) = \delta_{jk}, \quad |a_{ij}(\tau)| = 1. \quad (2.5)$$

We then have directly from (2.4)

$$\frac{\partial x_p(\tau)}{\partial X_q} = a_{pr}(\tau) \frac{\partial x_r(\tau)}{\partial X_q}. \quad (2.6)$$

We consider also that each of the vector fields $v_p^{(\alpha)}(\tau)$ ($\alpha=1, 2, \dots, \nu$) is subjected to the same rotation. Then their components $\bar{v}_p^{(\alpha)}(\tau)$ in the system x are given by

$$\bar{v}_p^{(\alpha)}(\tau) = a_{pq}(\tau) v_q^{(\alpha)}(\tau). \quad (2.7)$$

Suppose that if the tensor field $u_{i_1 i_2 \dots i_\mu}(\tau)$ is subjected to the same rotation, it has components $\bar{u}_{i_1 i_2 \dots i_\mu}(\tau)$ in the system x . Then,

$$\bar{u}_{i_1 i_2 \dots i_\mu} = a_{i_1 j_1} a_{i_2 j_2} \dots a_{i_\mu j_\mu} u_{j_1 j_2 \dots j_\mu}, \quad (2.8)$$

where the notation

$$a_{ij} = a_{ij}(t), \quad \bar{u}_{i_1 i_2 \dots i_\mu} = \bar{u}_{i_1 i_2 \dots i_\mu}(t) \quad (2.9)$$

is used.

Let x' be the coordinate system into which the coordinate system x is rotated by the rotation described by (2.4). From our assumption that the constitutive equation is unaltered by a simultaneous rotation of the reference system and of the physical system as a whole, we see that in the coordinate system x' the constitutive equation is (2.2). In the coordinate system x the constitutive equation may, of course, be written as

$$\bar{u}_{i_1 i_2 \dots i_\mu} = F_{i_1 i_2 \dots i_\mu} \left[\frac{\partial \bar{x}(\tau)}{\partial X_q}, \bar{v}_p^{(\alpha)}(\tau) \right]_{\tau=0}. \quad (2.10)$$

From (2.2), (2.8) and (2.10), we obtain

$$F_{i_1 i_2 \dots i_\mu} \left[\frac{\partial \bar{x}_p(\tau)}{\partial X_q}, \bar{v}_p^{(\alpha)}(\tau) \right] = a_{i_1 j_1} a_{i_2 j_2} \dots a_{i_\mu j_\mu} F_{j_1 j_2 \dots j_\mu} \left[\frac{\partial x_p(\tau)}{\partial X_q}, v_p^{(\alpha)}(\tau) \right]_{\tau=0} \quad (2.11)$$

This gives in implicit form the restriction imposed on the functionals $F_{i_1 i_2 \dots i_\mu}$ by invariance of the constitutive equation under simultaneous rotation of the physical system and the reference system. In the next section, we shall obtain these restrictions in explicit form.

3. Restrictions on the constitutive equation due to rotation in explicit form

We replace the functional dependence of $F_{i_1 i_2 \dots i_\mu}$ in (2.2), and hence in (2.11), on its arguments $\partial x_p(\tau)/\partial X_q$ and $v_p^{(\alpha)}(\tau)$, by polynomial dependence on the values of $\partial x_p(\tau)/\partial X_q$ and $v_p^{(\alpha)}(\tau)$ at $N+1$ discrete instants of time τ_P ($P=0, 1, 2, \dots, N$) in the interval 0 to t , where $\tau_0=t$. Thus,

$$u_{i_1 i_2 \dots i_\mu} = f_{i_1 i_2 \dots i_\mu} \left[\partial x_p(\tau_P)/\partial X_q, v_p^{(\alpha)}(\tau_P) \right], \quad (3.1)$$

where $f_{i_1 i_2 \dots i_\mu}$ are polynomial functions of the indicated arguments which satisfy the relation

$$f_{i_1 i_2 \dots i_\mu} \left[\partial \bar{x}_p(\tau_P)/\partial X_q, \bar{v}_p^{(\alpha)}(\tau_P) \right] = a_{i_1 j_1} a_{i_2 j_2} \dots a_{i_\mu j_\mu} f_{j_1 j_2 \dots j_\mu} \left[\partial x_p(\tau_P)/\partial X_q, v_p^{(\alpha)}(\tau_P) \right]. \quad (3.2)$$

This relation must be valid for all $a_{ij}(\tau)$ satisfying (2.5).

Let us suppose first that $a_{ij}(\tau) = \delta_{ij}$ at all instants of time except in a small time interval about $\tau_0 \neq t$. Then (3.2) becomes

$$\begin{aligned} f_{i_1 i_2 \dots i_\mu} \left[\frac{\partial x_p(\tau_0)}{\partial X_q}, \frac{\partial x_p(\tau_1)}{\partial X_q}, \dots, \frac{\partial \bar{x}_p(\tau_0)}{\partial X_q}, \dots, \frac{\partial x_p(\tau_N)}{\partial X_q}, \right. \\ \left. v_p^{(\alpha)}(\tau_0), v_p^{(\alpha)}(\tau_1), \dots, \bar{v}_p^{(\alpha)}(\tau_0), \dots, v_p^{(\alpha)}(\tau_N) \right] \\ = f_{i_1 i_2 \dots i_\mu} \left[\frac{\partial x_p(\tau_0)}{\partial X_q}, \frac{\partial x_p(\tau_1)}{\partial X_q}, \dots, \frac{\partial x_p(\tau_0)}{\partial X_q}, \dots, \frac{\partial x_p(\tau_N)}{\partial X_q}, \right. \\ \left. v_p^{(\alpha)}(\tau_0), v_p^{(\alpha)}(\tau_1), \dots, v_p^{(\alpha)}(\tau_0), \dots, v_p^{(\alpha)}(\tau_N) \right]. \end{aligned} \quad (3.3)$$

This implies that $f_{i_1 i_2 \dots i_\mu}$ is a polynomial scalar invariant of the $\nu+3$ vectors $\partial x_p(\tau_0)/\partial X_1$, $\partial x_p(\tau_0)/\partial X_2$, $\partial x_p(\tau_0)/\partial X_3$ and $v_p^{(\alpha)}(\tau_0)$ ($\alpha=1, 2, \dots, \nu$). It follows that $f_{i_1 i_2 \dots i_\mu}$ must depend on $\partial x_p(\tau_0)/\partial X_q$ and $v_p^{(\alpha)}(\tau_0)$ through the elements of an integrity basis for the ν vectors. Such an integrity basis is given by

$$\begin{aligned} G_{rs}(\tau_0), \quad V_r^{(\alpha)}(\tau_0), \quad v_p^{(\alpha)}(\tau_0) v_p^{(\beta)}(\tau_0), \\ [G(\tau_0)]^{\frac{1}{2}}, \quad e_{ijk} \frac{\partial x_i(\tau_0)}{\partial X_p} \frac{\partial x_j(\tau_0)}{\partial X_q} v_k^{(\alpha)}(\tau_0), \\ e_{ijk} \frac{\partial x_i(\tau_0)}{\partial X_p} v_j^{(\alpha)}(\tau_0) v_k^{(\beta)}(\tau_0), \quad e_{ijk} v_i^{(\alpha)}(\tau_0) v_j^{(\beta)}(\tau_0) v_k^{(\gamma)}(\tau_0), \end{aligned} \quad (3.4)$$

where the notation

$$G_{rs}(\tau) = \frac{\partial x_p(\tau)}{\partial X_r} \frac{\partial x_p(\tau)}{\partial X_s}, \quad G(\tau) = |G_{rs}(\tau)| \quad \text{and} \quad V_r^{(\alpha)}(\tau) = \frac{\partial x_p(\tau)}{\partial X_r} v_p^{(\alpha)}(\tau), \quad (3.5)$$

is used and $e_{ijk} = 1$ or -1 , accordingly as i, j, k is an even or odd permutation of $1, 2, 3$ and $e_{ijk} = 0$ otherwise.

In a manner similar to that employed in a somewhat similar problem by PIPKIN & RIVLIN [6], it can be shown that any polynomial in the quantities (3.4) may be expressed as a polynomial in

$$G_{rs}(\tau_Q), \quad [G(\tau_Q)]^{-\frac{1}{2}} \quad \text{and} \quad V_r^{(\alpha)}(\tau_Q). \quad (3.6)$$

Repeating this argument for each of the instants of time $\tau_1, \tau_2, \dots, \tau_N$, we see that $f_{i_1 i_2 \dots i_\mu}$ must be expressible as a polynomial $h_{i_1 i_2 \dots i_\mu}$ (say) in $\partial x_p / \partial X_q, v_p^{(\alpha)}, G_{pq}(\tau_P), [G(\tau_P)]^{-\frac{1}{2}}, V_p^{(\alpha)}(\tau_P), (P=1, 2, \dots, N; \alpha=1, 2, \dots, \nu)$, thus:

$$f_{i_1 i_2 \dots i_\mu} = h_{i_1 i_2 \dots i_\mu} \left[\frac{\partial x_p}{\partial X_q}, v_p^{(\alpha)}, G_{pq}(\tau_P), \{G(\tau_P)\}^{-\frac{1}{2}}, V_p^{(\alpha)}(\tau_P) \right], \quad (3.7)$$

where the notation $x_p = x_p(t)$ and $v_p^{(\alpha)} = v_p^{(\alpha)}(t)$ is used. Introducing (3.7) into (3.2) and taking $a_{ij}(\tau) = \delta_{ij}$ for all instants of time except in a small time interval about $\tau = t$, we obtain

$$h_{i_1 i_2 \dots i_\mu} \left[\frac{\partial \bar{x}_p}{\partial X_q}, \bar{v}_p^{(\alpha)} \right] = a_{i_1 j_1} a_{i_2 j_2} \dots a_{i_\mu j_\mu} h_{j_1 j_2 \dots j_\mu} \left[\frac{\partial x_p}{\partial X_q}, v_p^{(\alpha)} \right], \quad (3.8)$$

dependence of $h_{i_1 i_2 \dots i_\mu}$ on both sides of (3.8) on the remaining variables $G_{pq}(\tau_P), [G(\tau_P)]^{-\frac{1}{2}}$ and $V_p^{(\alpha)}(\tau_P)$ being understood. It has been shown in the previous paper [6] that a restriction of the form (3.8) implies that $h_{i_1 i_2 \dots i_\mu}$ must be expressible in terms of $\partial x_p / \partial X_q$ and $v_p^{(\alpha)}$ by

$$h_{i_1 i_2 \dots i_\mu} = \frac{\partial x_{i_1}}{\partial X_{j_1}} \frac{\partial x_{i_2}}{\partial X_{j_2}} \dots \frac{\partial x_{i_\mu}}{\partial X_{j_\mu}} \varphi_{j_1 j_2 \dots j_\mu}(G_{pq}, G^{-\frac{1}{2}}, V_p^{(\alpha)}), \quad (3.9)$$

where $\varphi_{j_1 j_2 \dots j_\mu}$ is a polynomial in the indicated arguments and in the arguments $G_{pq}(\tau_P), [G(\tau_P)]^{-\frac{1}{2}}$ and $V_p^{(\alpha)}(\tau_P), (P=1, 2, \dots, N)$. From this result together with (3.7) and (3.1), we obtain

$$u_{i_1 i_2 \dots i_\mu} = \frac{\partial x_{i_1}}{\partial X_{j_1}} \frac{\partial x_{i_2}}{\partial X_{j_2}} \dots \frac{\partial x_{i_\mu}}{\partial X_{j_\mu}} \varphi_{j_1 j_2 \dots j_\mu} [G_{pq}(\tau_P), \{G(\tau_P)\}^{-\frac{1}{2}}, V_p^{(\alpha)}(\tau_P)], \quad (3.10)$$

where $P=0, 1, 2, \dots, N$. Since we may express $\{G(\tau_P)\}^{-\frac{1}{2}}$ as a polynomial in $G_{pq}(\tau_P)$ to any desired degree of approximation, we may without serious loss of generality omit the arguments $\{G(\tau_P)\}^{-\frac{1}{2}}$ in (3.10). Now, allowing the number of instants N to increase indefinitely, we obtain

$$u_{i_1 i_2 \dots i_\mu} = \frac{\partial x_{i_1}}{\partial X_{j_1}} \frac{\partial x_{i_2}}{\partial X_{j_2}} \dots \frac{\partial x_{i_\mu}}{\partial X_{j_\mu}} \Phi_{j_1 j_2 \dots j_\mu} \left[G_{pq}(\tau), V_p^{(\alpha)}(\tau) \right]_{\tau=0}, \quad (3.11)$$

where $\Phi_{j_1 j_2 \dots j_\mu}$ is a functional of its arguments.

4. Restrictions on the constitutive equation due to material symmetry

We shall now consider the further restrictions imposed on the constitutive equation (3.11) by any symmetry which the material may possess in its initial state, in which it is undeformed and $v_p^{(\alpha)} = 0$. The constitutive equation must be

form-invariant under the transformations from one rectangular Cartesian coordinate system to every equivalent rectangular Cartesian coordinate system. If x and $\bar{x}$ are equivalent coordinate systems related by

$$\bar{x}_i(\tau) = s_{ij} x_j(\tau), \quad (4.1)$$

where

$$s_{ij} s_{ik} = s_{ji} s_{ki} = \delta_{jk}, \quad (4.2)$$

and the constitutive equation takes the form (3.11) in the system x , then it takes the form

$$\bar{u}_{i_1 i_2 \dots i_\mu} = \frac{\partial \bar{x}_{i_1}}{\partial X_{j_1}} \frac{\partial \bar{x}_{i_2}}{\partial X_{j_2}} \dots \frac{\partial \bar{x}_{i_\mu}}{\partial X_{j_\mu}} \Phi_{j_1 j_2 \dots j_\mu} \left[\bar{G}_{pq}(\tau), \bar{V}_p^{(\alpha)}(\tau) \right]_{\tau=0}^t \quad (4.3)$$

in the system $\bar{x}$, where $\bar{x}_i$ is given by (4.1),

$$\begin{aligned} \bar{X}_i &= s_{ij} X_j, & \bar{G}_{pq}(\tau) &= s_{pr} s_{qs} G_{rs}(\tau), \\ \bar{V}_p^{(\alpha)} &= s_{pq} V_p^{(\alpha)} \quad \text{and} \quad \bar{u}_{i_1 i_2 \dots i_\mu} &= s_{i_1 j_1} s_{i_2 j_2} \dots s_{i_\mu j_\mu} u_{j_1 j_2 \dots j_\mu}. \end{aligned} \quad (4.4)$$

From (3.11), (4.3) and (4.4), we obtain

$$\begin{aligned} s_{i_1 j_1} s_{i_2 j_2} \dots s_{i_\mu j_\mu} \frac{\partial x_{j_1}}{\partial X_{k_1}} \frac{\partial x_{j_2}}{\partial X_{k_2}} \dots \frac{\partial x_{j_\mu}}{\partial X_{k_\mu}} \Phi_{k_1 k_2 \dots k_\mu} \left[G_{pq}(\tau), V_p^{(\alpha)}(\tau) \right]_{\tau=0}^t \\ = \frac{\partial \bar{x}_{i_1}}{\partial \bar{X}_{j_1}} \frac{\partial \bar{x}_{i_2}}{\partial \bar{X}_{j_2}} \dots \frac{\partial \bar{x}_{i_\mu}}{\partial \bar{X}_{j_\mu}} \Phi_{j_1 j_2 \dots j_\mu} \left[\bar{G}_{pq}(\tau), \bar{V}_p^{(\alpha)}(\tau) \right]_{\tau=0}^t. \end{aligned} \quad (4.5)$$

This restriction on the form of the functional $\Phi_{i_1 i_2 \dots i_\mu}$ must be valid for the transformations relating the coordinate system x with each of the equivalent coordinate systems. We can obtain this restriction in a more explicit form in the following way.

Let $\psi_i^{(\beta)}$ and $\bar{\psi}_i^{(\beta)}$ ($\beta=1, 2, \dots, \mu$) be the components in the systems x and $\bar{x}$ respectively of μ vectors. Then

$$\psi_i^{(\beta)} = s_{ji} \bar{\psi}_j^{(\beta)}. \quad (4.6)$$

Multiplying (4.5) throughout by $\bar{\psi}_{i_1}^{(1)} \bar{\psi}_{i_2}^{(2)} \dots \bar{\psi}_{i_\mu}^{(\mu)}$, employing (4.6) and the notation

$$\Psi_i^{(\beta)} = \psi_p^{(\beta)} \frac{\partial x_p}{\partial X_i}, \quad \bar{\Psi}_i^{(\beta)} = \bar{\psi}_p^{(\beta)} \frac{\partial \bar{x}_p}{\partial \bar{X}_i}, \quad (4.7)$$

we obtain

$$\begin{aligned} \Psi_{k_1}^{(1)} \Psi_{k_2}^{(2)} \dots \Psi_{k_\mu}^{(\mu)} \Phi_{k_1 k_2 \dots k_\mu} \left[G_{pq}(\tau), V_p^{(\alpha)}(\tau) \right]_{\tau=0}^t \\ = \bar{\Psi}_{k_1}^{(1)} \bar{\Psi}_{k_2}^{(2)} \dots \bar{\Psi}_{k_\mu}^{(\mu)} \Phi_{k_1 k_2 \dots k_\mu} \left[\bar{G}_{pq}(\tau), \bar{V}_p^{(\alpha)}(\tau) \right] \\ = F \text{ (say)}. \end{aligned} \quad (4.8)$$

With this definition of F , we note, from (4.7) and (3.11), that

$$u_{i_1 i_2 \dots i_\mu} = \frac{\partial F}{\partial \psi_{i_1}^{(1)} \partial \psi_{i_2}^{(2)} \dots \partial \psi_{i_\mu}^{(\mu)}}. \quad (4.9)$$

We now assume that, in our initial equation (2.2), the functional $F_{i_1 i_2 \dots i_\mu}$ is a continuous functional of $\partial x_p(\tau)/\partial X_q$ and $v_p^{(\alpha)}(\tau)$ ($\alpha=1, 2, \dots, r$), over the compact aggregate of these functions which are continuous in the range $0 \leq \tau \leq t$, and is particularly dependent on the values of these functions at the instant t . It follows that the functional $\Phi_{i_1 i_2 \dots i_\mu}$ is a continuous functional of $G_{pq}(\tau)$ and

$V_p^{(\alpha)}(\tau)$, over the compact aggregate of these functions which are continuous in the range $0 \leq \tau \leq t$, and is, in general, particularly dependent on the values G_{pq} and $V_p^{(\alpha)}$ of these functions at the instant t . We assume sufficient regularity so that $\Phi_{i_1 i_2 \dots i_\mu}$ may be expressed as a polynomial in G_{pq} and $V_p^{(\alpha)}$, the coefficients of which are continuous functionals of $G_{pq}(\tau)$ and $V_p^{(\alpha)}(\tau)$, over the compact aggregate of these functions which are continuous in the range $0 \leq \tau \leq t$.

We shall denote the Fourier half-range cosine coefficients of $G_{pq}(\tau)$ by $\gamma_{pq}^{(P)}$ ($P=0, 1, 2, \dots$) and those of $V_p^{(\alpha)}(\tau)$ by ${}^{(P)}\omega_p^{(\alpha)}$ ($P=0, 1, 2, \dots$), where

$$\begin{aligned}\gamma_{pq}^{(P)} &= \frac{2}{t} \int_0^t G_{pq}(\tau) \cos \frac{P\pi\tau}{t} d\tau \quad (P > 0), \\ \gamma_{pq}^{(0)} &= \frac{1}{t} \int_0^t G_{pq}(\tau) d\tau, \\ {}^{(P)}\omega_p^{(\alpha)} &= \frac{2}{t} \int_0^t V_p^{(\alpha)}(\tau) \cos \frac{P\pi\tau}{t} d\tau \quad (P > 0) \\ \text{and} \\ {}^{(0)}\omega_p^{(\alpha)} &= \frac{1}{t} \int_0^t V_p^{(\alpha)}(\tau) d\tau.\end{aligned}\tag{4.10}$$

Then each continuous functional in the expression for $\Phi_{i_1 i_2 \dots i_\mu}$ may be approximated by a polynomial

$$P^{(N)}(\gamma_{pq}^{(0)}, \gamma_{pq}^{(1)}, \dots, \gamma_{pq}^{(N)}, {}^{(0)}\omega_p^{(\alpha)}, {}^{(1)}\omega_p^{(\alpha)}, \dots, {}^{(N)}\omega_p^{(\alpha)})$$

which tends uniformly to the functional as $N \rightarrow \infty$. It follows that $\Phi_{i_1 i_2 \dots i_\mu}$ may be expressed to any desired approximation as a polynomial $P_{i_1 i_2 \dots i_\mu}^*$ in

$$G_{pq}, V_p^{(\alpha)}, \gamma_{pq}^{(0)}, \gamma_{pq}^{(1)}, \dots, \gamma_{pq}^{(N)}, {}^{(0)}\omega_p^{(\alpha)}, {}^{(1)}\omega_p^{(\alpha)}, \dots, {}^{(N)}\omega_p^{(\alpha)}.$$

Defining $\bar{\gamma}_{pq}^{(P)}$ and ${}^{(P)}\bar{\omega}_p^{(\alpha)}$ in terms of $\bar{G}_{pq}(\tau)$ and $\bar{V}_p^{(\alpha)}(\tau)$ in the same way as $\gamma_{pq}^{(P)}$ and ${}^{(P)}\omega_p^{(\alpha)}$ are defined in terms of $G_{pq}(\tau)$ and $V_p^{(\alpha)}(\tau)$, we have

$$\begin{aligned}\bar{\gamma}_{pq}^{(P)} &= s_{pq} s_{qs} \gamma_{rs}^{(P)} \\ {}^{(P)}\bar{\omega}_p^{(\alpha)} &= s_{pq} {}^{(P)}\omega_q^{(\alpha)}.\end{aligned}\tag{4.11}$$

We note also from (4.1), (4.4), (4.6) and (4.7) that

$$\bar{\Psi}_p^{(\beta)} = s_{pq} \Psi_q^{(\beta)}.\tag{4.12}$$

Replacing $\Phi_{i_1 i_2 \dots i_\mu}$ in (4.8) by $P_{i_1 i_2 \dots i_\mu}^*$, it follows that F is a polynomial scalar invariant, under the group of transformations characterising the symmetry of the material, of the $\mu + \nu(N+2)$ vectors $\Psi_p^{(\alpha)}$ ($\alpha=1, 2, \dots, \mu$), $V_p^{(\alpha)}$ ($\alpha=1, 2, \dots, \nu$) and ${}^{(P)}\omega_p^{(\alpha)}$ ($\alpha=1, 2, \dots, \nu$; $P=0, 1, 2, \dots, N$) and the $N+2$ symmetric tensors G_{pq} and $\gamma_{pq}^{(P)}$ ($P=0, 1, 2, \dots, N$). It follows that F must be expressible as a polynomial in the elements of an integrity basis for these vectors and tensors and we note that F is multilinear in the vectors $\Psi_p^{(\alpha)}$ ($\alpha=1, 2, \dots, \mu$), so that we need not consider those elements of the integrity basis which are non-linear in any one of the vectors $\Psi_p^{(\alpha)}$.

5. Isotropic materials possessing a center of symmetry

Following the argument of the previous section, we see that if, prior to time $\tau=0$, the material is isotropic and possesses a center of symmetry, F must be expressible as a polynomial in the elements of an integrity basis, under the full orthogonal group of transformations, of the $\mu + \nu(N+2)$ vectors $\Psi_p^{(\alpha)}$ ($\alpha=1, 2, \dots, \mu$), $V_p^{(\alpha)}$ ($\alpha=1, 2, \dots, \nu$) and $^{(P)}\omega_p^{(\alpha)}$ ($\alpha=1, 2, \dots, \nu$; $P=0, 1, 2, \dots, N$) and the $N+2$ symmetric tensors G_{pq} and $\gamma_{pq}^{(P)}$ ($P=0, 1, 2, \dots, N$). Using the notation Ψ_α , V_α and $\omega_\alpha^{(P)}$ for the vectors $\Psi_p^{(\alpha)}$, $V_p^{(\alpha)}$ and $^{(P)}\omega_p^{(\alpha)}$ respectively and the notation $\mathbf{G}$ and $\boldsymbol{\gamma}_P$ for the symmetric matrices $\|G_{pq}\|$ and $\|\gamma_{pq}^{(P)}\|$ respectively and omitting elements which are non-linear in any one of the vectors Ψ_α , such an integrity basis, although not in general an irreducible one, is

$$\begin{aligned} V_\alpha \Pi \Psi_\beta & \quad (\alpha=1, 2, \dots, \nu; \beta=1, 2, \dots, \mu) \\ \omega_\alpha^{(P)} \Pi \Psi_\beta & \quad (\alpha=1, 2, \dots, \nu; \beta=1, 2, \dots, \mu; P=0, 1, 2, \dots, N) \\ \Psi_\alpha \Pi \Psi_\beta & \quad (\alpha, \beta=1, 2, \dots, \nu; \alpha \neq \beta) \\ V_\alpha \Pi V_\beta, \quad \omega_\alpha^{(P)} \Pi \omega_\beta^{(Q)}, \quad \omega_\alpha^{(P)} \Pi V_\beta & \quad (\alpha, \beta=1, 2, \dots, \nu; P, Q=0, 1, 2, \dots, N) \end{aligned} \quad (5.1)$$

and $\text{tr } \Pi$,

where Π are all possible matrix products, including the unit matrix, formed from the matrices $\mathbf{G}$ and $\boldsymbol{\gamma}_P$ ($P=0, 1, 2, \dots, N$).

We denote the μ^{th} order tensor $u_{i_1 i_2 \dots i_\mu}$ by $\mathbf{u}$ and the matrix $\|\partial x_i / \partial X_j\|$ by $\mathbf{c}$. Also, we denote the transpose of $\mathbf{c}$ by $\mathbf{c}'$. Then, bearing in mind the relation (4.9) and the definition of Ψ_α contained in (4.7), we see that $\mathbf{u}$ must be expressible as a sum of outer products of order μ formed from vectors of the types $V_\alpha \Pi \mathbf{c}'$ and $\omega_\alpha^{(P)} \Pi \mathbf{c}'$ and the matrices $\mathbf{c} \Pi \mathbf{c}'$, the coefficients of these terms being polynomials in invariants of the types $V_\alpha \Pi V_\beta$, $\omega_\alpha^{(P)} \Pi \omega_\beta^{(Q)}$, $\omega_\alpha^{(P)} \Pi V_\beta$ ($\alpha, \beta=1, 2, \dots, \nu$; $P, Q=0, 1, 2, \dots, N$) and $\text{tr } \Pi$.

Consequently, we must have

$$u_{i_1 i_2 \dots i_\mu} = \frac{\partial x_{i_1}}{\partial X_{j_1}} \frac{\partial x_{i_2}}{\partial X_{j_2}} \dots \frac{\partial x_{i_\mu}}{\partial X_{j_\mu}} \Phi_{j_1 j_2 \dots j_\mu}, \quad (5.2)$$

where $\Phi_{j_1 j_2 \dots j_\mu}$ is a sum of outer products of order μ formed from vectors of the types $V_\alpha \Pi$ and $\omega_\alpha^{(P)} \Pi$ and matrices of the type Π , the coefficients of these outer products being polynomials in invariants of the types $V_\alpha \Pi V_\beta$, $\omega_\alpha^{(P)} \Pi \omega_\beta^{(Q)}$, $V_\beta \Pi \omega_\alpha^{(P)}$ and $\text{tr } \Pi$. We note from (4.10) that matrices of the type Π and vectors of the types $V_\alpha \Pi$ and $\omega_\alpha^{(P)} \Pi$ may be expressed in the forms

$$\begin{aligned} \Pi &= \int_0^t \int_0^t \dots \int_0^t \chi(t, \tau_1, \tau_2, \dots, \tau_R) \times \\ &\quad \times \Pi^* [\mathbf{G}, \mathbf{G}(\tau_1), \mathbf{G}(\tau_2), \dots, \mathbf{G}(\tau_R)] d\tau_1 d\tau_2 \dots d\tau_R, \\ V_\alpha \Pi &= V_\alpha \int_0^t \int_0^t \dots \int_0^t \chi(t, \tau_1, \tau_2, \dots, \tau_R) \times \\ &\quad \times \Pi^* [\mathbf{G}, \mathbf{G}(\tau_1), \mathbf{G}(\tau_2), \dots, \mathbf{G}(\tau_R)] d\tau_1 d\tau_2 \dots d\tau_R, \\ \omega_\alpha^{(P)} \Pi &= \int_0^t \int_0^t \dots \int_0^t \chi(t, \tau_0, \tau_1, \tau_2, \dots, \tau_R) \times \\ &\quad \times V_\alpha(\tau_0) \Pi^* [\mathbf{G}, \mathbf{G}(\tau_1), \mathbf{G}(\tau_2), \dots, \mathbf{G}(\tau_R)] d\tau_0 d\tau_1 \dots d\tau_R, \end{aligned} \quad (5.3)$$

where $\mathbf{\Pi}^*$ is a matrix product formed from the matrices $\mathbf{G}$, $\mathbf{G}(\tau_1), \dots, \mathbf{G}(\tau_R)$ which is linear in each of the matrices $\mathbf{G}(\tau_1), \dots, \mathbf{G}(\tau_R)$, and $\chi(t, \tau_1, \tau_2, \dots, \tau_R)$ is a continuous function of its arguments.

In a similar manner we can express invariants of the types $\text{tr } \mathbf{\Pi}$, $\mathbf{V}_\alpha \mathbf{\Pi} \mathbf{V}_\beta$, $\omega_\alpha^{(P)} \mathbf{\Pi} \omega_\beta^{(Q)}$ and $\mathbf{V}_\beta \mathbf{\Pi} \omega_\alpha^{(P)}$ in the forms

$$\begin{aligned} \text{tr } \mathbf{\Pi} &= \int_0^t \int_0^t \cdots \int_0^t \chi(t, \tau_1, \tau_2, \dots, \tau_R) \times \\ &\quad \times \text{tr } \mathbf{\Pi}^* [\mathbf{G}, \mathbf{G}(\tau_1), \mathbf{G}(\tau_2), \dots, \mathbf{G}(\tau_R)] d\tau_1 d\tau_2 \dots d\tau_R, \\ \mathbf{V}_\alpha \mathbf{\Pi} \mathbf{V}_\beta &= \int_0^t \int_0^t \cdots \int_0^t \chi(t, \tau_1, \tau_2, \dots, \tau_R) \times \\ &\quad \times \mathbf{V}_\alpha \mathbf{\Pi}^* [\mathbf{G}, \mathbf{G}(\tau_1), \mathbf{G}(\tau_2), \dots, \mathbf{G}(\tau_R)] \mathbf{V}_\beta d\tau_1 d\tau_2 \dots d\tau_R, \\ \omega_\alpha^{(P)} \mathbf{\Pi} \omega_\beta^{(Q)} &= \int_0^t \int_0^t \cdots \int_0^t \chi(t, \tau_0, \tau_1, \dots, \tau_R, \tau_{R+1}) \times \\ &\quad \times \mathbf{V}_\alpha(\tau_0) \mathbf{\Pi}^* [\mathbf{G}, \mathbf{G}(\tau_1), \mathbf{G}(\tau_2), \dots, \mathbf{G}(\tau_R)] \mathbf{V}_\beta(\tau_{R+1}) d\tau_0 d\tau_1 \dots d\tau_R d\tau_{R+1}, \\ \mathbf{V}_\beta \mathbf{\Pi} \omega_\alpha^{(P)} &= \int_0^t \int_0^t \cdots \int_0^t \chi(t, \tau_1, \tau_2, \dots, \tau_R, \tau_{R+1}) \times \\ &\quad \times \mathbf{V}_\beta \mathbf{\Pi}^* [\mathbf{G}, \mathbf{G}(\tau_1), \mathbf{G}(\tau_2), \dots, \mathbf{G}(\tau_R)] \mathbf{V}_\alpha(\tau_{R+1}) d\tau_1 d\tau_2 \dots d\tau_R d\tau_{R+1}. \end{aligned} \quad (5.4)$$

Consequently $u_{i_1 i_2 \dots i_\mu}$ may be expressed in the form (5.2), where $\Phi_{j_1 j_2 \dots j_\mu}$ may be expressed as a sum of outer products of order μ formed from expressions of the types given on the right-hand sides of equations (5.3), with coefficients which are polynomials in expressions of the types given on the right-hand sides of equations (5.4).

In the expressions (5.3) and (5.4), we have so far allowed $\mathbf{\Pi}^*$ to be any matrix products formed from $\mathbf{G}$, $\mathbf{G}(\tau_1), \dots, \mathbf{G}(\tau_R)$ which are linear in each of the matrices $\mathbf{G}(\tau_1), \dots, \mathbf{G}(\tau_R)$. We may in fact limit the forms taken by these matrix products to a considerable extent and this will be done in a paper which is at present in course of preparation.

6. A class of physical problems for an undeformed body

If the body undergoes no deformation, the constitutive equation (2.2) takes the form

$$u_{i_1 i_2 \dots i_\mu} = F_{i_1 i_2 \dots i_\mu} \left[v_p^{(\alpha)}(\tau) \right]. \quad (6.1)$$

We shall assume that the functionals $F_{i_1 i_2 \dots i_\mu}$ are continuous functionals of $v_p^{(\alpha)}(\tau)$ over the compact aggregate of these functions which are continuous in the range $0 \leq \tau \leq t$ and are particularly dependent on the values of these functions at $\tau = t$.

Now, suppose we limit ourselves to the class of problems in which each of the vectors $v_p^{(\alpha)}(\tau)$ varies in some prescribed manner with time, so that

$$v_p^{(\alpha)} = f^{(\alpha)}(\tau) w_p^{(\alpha)}, \quad (6.2)$$

where the functions $f^{(\alpha)}(\tau)$ are specified and the vectors $w_p^{(\alpha)}$ are independent of time. Then, introducing (6.2) into (6.1), we see that

$$u_{i_1 i_2 \dots i_\mu} = f_{i_1 i_2 \dots i_\mu}(w_p^{(\alpha)}, t), \quad (6.3)$$

where $f_{i_1 i_2 \dots i_\mu}$ are ordinary functions of the indicated variables, determined by the functionals $F_{i_1 i_2 \dots i_\mu}$ and the functions $f^{(\alpha)}(\tau)$. We shall assume for simplicity that $f_{i_1 i_2 \dots i_\mu}$ are polynomial functions of $w_p^{(\alpha)}$, or may be represented with sufficient accuracy by such polynomials.

By taking $\partial x_i(\tau)/\partial X_j = G_{ij}(\tau) = \delta_{ij}$ in (3.11) and bearing in mind the definition of $V_p^{(\alpha)}(\tau)$ given in (3.5), we see that the assumption that the constitutive equation is unaltered by a simultaneous rotation of the reference system and the physical system imposes no restriction on the constitutive equation in the case when the body does not undergo deformation*. Consequently, we may pass immediately to the consideration of the effect of symmetry of the material in restricting the form of the constitutive equation.

These restrictions arise, as previously discussed, from the fact that the constitutive equation must be form-invariant under the group of transformations $\{s\}$ relating a complete set of equivalent coordinate systems in the material. Then, following an argument similar to that employed in § 4 and in previous papers, we obtain

$$u_{i_1 i_2 \dots i_\mu} = \frac{\partial^\mu F}{\partial \psi_{i_1}^{(1)} \partial \psi_{i_2}^{(2)} \dots \partial \psi_{i_\mu}^{(\mu)}}, \quad (6.4)$$

where F is a polynomial scalar invariant under the group of transformations $\{s\}$ of the v vectors $w_i^{(\alpha)}$ ($\alpha = 1, 2, \dots, v$) and the μ vectors $\psi_i^{(\beta)}$ ($\beta = 1, 2, \dots, \mu$); F is multilinear in the vectors $\psi_i^{(\beta)}$ and is also a single-valued function of t . Apart from the dependence on t , this is the form which would be obtained for $u_{i_1 i_2 \dots i_\mu}$ if we had considered a constitutive equation relating a tensor $u_{i_1 i_2 \dots i_\mu}$ and v vectors $w_p^{(\alpha)}$ for a material of the same symmetry, for which the relation is not time-dependent. Such constitutive equations arise as special cases, in which the deformation is taken as zero, of the constitutive equations discussed in the previous paper by PIPKIN & RIVLIN [6]. In that paper, the restrictions on the form of the constitutive equations imposed by the symmetry of the material are obtained in explicit form for a number of types of symmetry. These results may be readily applied to the case under consideration at present by introducing the fact that the body is undeformed and inserting dependence of $u_{i_1 i_2 \dots i_\mu}$ on t . For example, in the case when $\mu = 2$, and the material considered is isotropic and possesses a center of symmetry, it is seen from the results obtained in [6] that if we take as our starting point a constitutive equation of the form

$$u_{ij} = f_{ij}(w_i^{(\alpha)}), \quad (6.5)$$

where f_{ij} are polynomials, then u_{ij} must be expressible in the form

$$u_{ij} = \sum_{\alpha, \beta=1}^v A_{\alpha\beta} w_i^{(\alpha)} w_j^{(\beta)} + B \delta_{ij}, \quad (6.6)$$

* If we examine the effect of this assumption on the form of equation (6.3) directly, we find that it is equivalent to the restriction that the material is initially isotropic, but does not necessarily possess a center of symmetry. This apparent paradox vanishes if we realise that when the body and associated vector fields are rotated, the constitutive equation in the unrotated coordinate system x will not have the same form as it does originally, unless we include, as dependent variables in the initial equation, three mutually perpendicular vectors which rotate with the body and serve to define the directions of the symmetry axes relative to the reference system. If the deformation gradients occur in the constitutive equation, they provide these vectors.

where $A_{\alpha\beta}$ and B are polynomials in $w_p^{(\alpha)} w_p^{(\beta)}$. It follows immediately that if we take (6.3) as our starting point, then u_{ij} is expressible in the form (6.6), where $A_{\alpha\beta}$ are polynomials in $w_p^{(\alpha)} w_p^{(\beta)}$ and functions of t . It may be noted that the same result is obtainable from the results of § 5 of the present paper by taking $\mathbf{G}(\tau) = \mathbf{I}$ and introducing the expressions (6.2) for $v_p^{(\alpha)}$.

7. A class of physical problems for a body which undergoes small deformations

If we take as our starting point the constitutive equation (2.2), then we have seen in § 3 that the assumption that it is unaltered by a simultaneous rotation of the reference system and the physical system implies that it can be expressed in the form (3.11).

We shall now make the assumption that the deformation to which the body is subjected is sufficiently small, so that the displacement gradients may be neglected in comparison with unity. Let $u_i(\tau)$ be the displacement components in the coordinate system x undergone at time τ by a generic particle of the body which is at X_i prior to time $\tau = 0$. Then,

$$x_i(\tau) = X_i + u_i(\tau). \quad (7.1)$$

Introducing (7.1) into the definition (3.5) of $G_{pq}(\tau)$ and neglecting terms of second degree in the displacement gradients in comparison with those of the first degree, we have

$$G_{pq}(\tau) - \delta_{pq} = 2e_{pq}(\tau), \quad (7.2)$$

where

$$e_{pq}(\tau) = \frac{1}{2} \left(\frac{\partial u_p(\tau)}{\partial X_q} + \frac{\partial u_q(\tau)}{\partial X_p} \right). \quad (7.3)$$

Also, from (3.5), we see that neglecting $\partial u_p(\tau)/\partial X_q$ in comparison with unity, we have

$$V_r^{(\alpha)}(\tau) = v_r^{(\alpha)}(\tau). \quad (7.4)$$

We now limit ourselves to the class of problems in which the displacement vector $u_p(\tau)$ and each of the vectors $v_i^{(\alpha)}(\tau)$ varies in some prescribed manner with time so that

$$\begin{aligned} u_p(\tau) &= g(\tau) U_p \\ v_p^{(\alpha)}(\tau) &= f^{(\alpha)}(\tau) w_p^{(\alpha)}, \end{aligned} \quad (7.5)$$

where the functions $g(\tau)$ and $f^{(\alpha)}(\tau)$ are specified and the vectors U_p and $w_p^{(\alpha)}$ are independent of time. Then, introducing the first of the relations (7.5) into (7.3) and the relation so obtained into (7.2), we have

$$G_{pq}(\tau) - \delta_{pq} = 2g(\tau) E_{pq}, \quad (7.6)$$

where

$$E_{pq} = \frac{1}{2} \left(\frac{\partial U_p}{\partial X_q} + \frac{\partial U_q}{\partial X_p} \right). \quad (7.7)$$

Using (7.4), the second of the relations (7.5) and the relation (7.6) in the functional $\Phi_{j_1 j_2 \dots j_\mu}$ occurring in (3.11), we see that it can be expressed as an ordinary

function of $w_p^{(\alpha)}$ and E_{pq} and of t , thus:

$$\Phi_{i_1 i_2 \dots i_\mu} \left[G_{pq}(\tau), V_p^{(\alpha)}(\tau) \right]_{\tau=0}^t = f_{i_1 i_2 \dots i_\mu} (E_{pq}, w_p^{(\alpha)}, t). \quad (7.8)$$

Introducing this result into (3.11), using (7.1) and neglecting terms of the first degree in $\partial u_p / \partial X_q$ in comparison with unity, we obtain

$$u_{i_1 i_2 \dots i_\mu} = f_{i_1 i_2 \dots i_\mu} (E_{pq}, w_p^{(\alpha)}, t). \quad (7.9)$$

The restrictions imposed on the form of this constitutive equation by symmetry of the material may be developed in a manner similar to that employed in the previous cases discussed. If the group of transformations characterising the symmetry of the material is $\{s\}$, then

$$u_{i_1 i_2 \dots i_\mu} = \frac{\partial^\mu F}{\partial \psi_{i_1}^{(1)} \partial \psi_{i_2}^{(2)} \dots \partial \psi_{i_\mu}^{(\mu)}}, \quad (7.10)$$

where F is a polynomial scalar invariant under the group of transformations $\{s\}$ of the ν vectors $w_i^{(\alpha)}$ ($\alpha = 1, 2, \dots, \nu$), the μ vectors $\psi_i^{(\beta)}$ ($\beta = 1, 2, \dots, \mu$) and the symmetric second-order tensor E_{pq} ; F is multilinear in the vectors $\psi_i^{(\beta)}$ ($\beta = 1, 2, \dots, \mu$) and is also a single-valued function of t .

For particular symmetries, we can again obtain explicit results, from those obtained by PIPKIN & RIVLIN [6] in the case when no time dependence is involved, by making the approximation that $\partial u_p / \partial X_q$ is small compared with unity and introducing the dependence on t .

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References

- [1] GREEN, A. E., & R. S. RIVLIN: Arch. Rational Mech. Anal. **1**, 1 (1957).
- [2] GREEN, A. E., R. S. RIVLIN & A. J. M. SPENCER: Arch. Rational Mech. Anal. **3**, 82 (1959).
- [3] GREEN, A. E., & R. S. RIVLIN: Brown Univ. Technical Report C 11-17, Contract Nonr-562(10).
- [4] NOLL, W.: Carnegie Inst. Technical Report No. 17, to Air Force Office Scientific Research 1957.
- [5] NOLL, W.: Arch. Rational Mech. Anal. **2**, 197 (1958).
- [6] PIPKIN, A. C., & R. S. RIVLIN: Arch. Rational Mech. Anal. **4**, 129 (1959).
- [7] RIVLIN, R. S.: Arch. Rational Mech. Anal. **3**, 304 (1959).

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